UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 9 June, 2006 9 to 12

PAPER 72

PHYSICAL COSMOLOGY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) The deceleration parameter is defined by

$$q_0 \equiv -\frac{\ddot{a}_0}{a_0 H_0^2},$$

where a_0 is the scale factor today and H_0 the Hubble constant. Prove that in a matter dominated Universe ($\Lambda = 0$)

$$\Omega_{m,0} = 2q_0,$$

where $\Omega_{m,0}$ is the ratio of the matter density today to that required for a flat universe.

(ii) Hence show that:

$$\left(\frac{\dot{a}}{a_0}\right)^2 = H_0^2 \left[1 - 2q_0 + 2q_0 \frac{a_0}{a}\right].$$

(iii) Show that the solutions to the equation at (ii) are as follows:

(a) $q_0 > 1/2$

$$H_0 \cdot t = q_0 (2q_0 - 1)^{-3/2} \left[\theta - \sin\theta\right],$$

where the development angle θ is defined by

$$1 - \cos\theta = \left(\frac{2q_0 - 1}{q_0}\right) \frac{a}{a_0}.$$

(b) $q_0 = 1/2$

$$H_0 \cdot t = \frac{2}{3}\left(\frac{a}{a_0}\right)^{3/2}.$$

(c) $0 < q_0 < 1/2$

$$H_0 \cdot t = q_0 (1 - 2q_0)^{-3/2} \left[\sinh\psi - \psi\right],$$

where

$$\cosh\psi - 1 = \left(\frac{1 - 2q_0}{q_0}\right) \frac{a}{a_0}.$$

You may use natural units, whereby $c = 1$, throughout.

2 (i) From the relation between redshift and scale factor:

$$1 + z(t_0, t_e) \equiv \frac{a(t_0)}{a(t_e)}$$

where t_0 is the time of observation and t_e is the time when a photon was emitted (or absorbed), show that the measured redshift of a distant astronomical source varies with t_0 as:

$$\dot{z} = \frac{dz}{dt_0} = H(t_0)\,(1+z) - H(t_e)$$

where $H(t)$ is the Hubble parameter at time t.

(ii) Discuss, with the aid of appropriate sketches, the behaviour of the function $\dot{z} = f(z)$ in flat cosmologies with:

(a) $\Omega_{m,0} = 1$, $\Omega_{\Lambda,0} = 0$;

(b) $\Omega_{m,0} = 0$, $\Omega_{\Lambda,0} = 1$;

(c) $\Omega_{m,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$.

Does the redshift at which $\dot{z} = 0$ depend on the present-day expansion rate?

(iii) Using the relation

$$\frac{d\lambda}{\lambda} = \frac{dv}{c}$$

where $d\lambda$ and dv are small intervals of wavelength and velocity, and assuming an Einstein-de Sitter cosmology ($\Omega_{m,0} = 1$; $\Omega_{\Lambda,0} = \Omega_{k,0} = 0$), estimate the velocity shift between two spectra of the same extragalactic source at $z = 3$ recorded at two epochs separated by 10 years. You can assume that the source has zero peculiar velocity relative to the cosmic expansion.

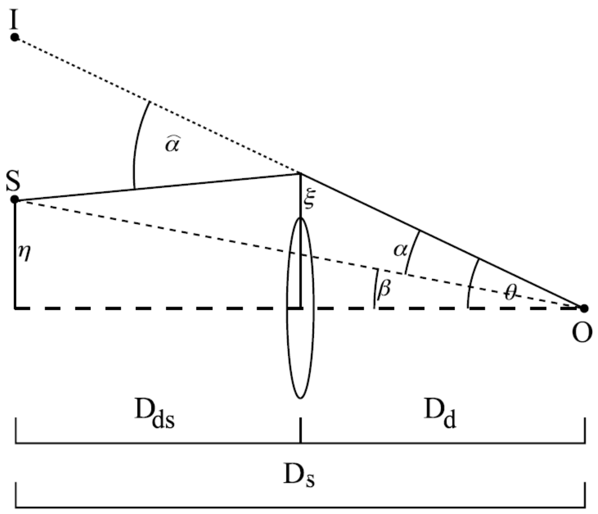
Note that:

(1) $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}} \simeq 2 \times 10^{-18}\,\mathrm{s^{-1}}$;

(2) 1 year $\simeq 3 \times 10^7$ s.

(iv) Comment on the feasibility of using observations of absorption lines in the Lyman alpha forest to measure directly the cosmic expansion.

3 The figure shows a light ray propagating from the point source S, being deflected by an angle $\hat{\alpha}$ by the gravitational potential of a lens L, and reaching the observer at O. The transverse distance of S from the optical axis connecting O and L is η, while ξ is the transverse distance from the optical axis of the light ray at the point of deflection. The angular separations of the source and image I from the optical axis, as seen by the observer, are β and θ respectively. The angular diameter distances between the observer and the lens, the observer and the source, and the lens and the source are, respectively, $D_{\rm d}$, $D_{\rm s}$, and $D_{\rm ds}$.



(i) Derive the relationship between $\hat{\alpha}$, η, ξ, $D_{\rm d}$, $D_{\rm s}$, and $D_{\rm ds}$ which must be satisfied for the light ray to reach the observer at O. Similarly, derive a relationship between $\hat{\alpha}$ and the scaled deflection angle α.

(ii) Use the relationships derived at (i) to verify the one-dimensional lens equation: $\beta = \theta - \alpha(\theta)$. Hence, given that for a point-mass lens:

$$\hat{\alpha} = \frac{4GM}{c^2\xi}$$

use the lens equation to solve for the Einstein radius $\theta_{\rm E}$ in terms of the angular diameter distances, and for the image positions θ in terms of $\theta_{\rm E}$ and β.

(iii) Given that the magnification of an image is

$$\mu = \frac{\theta}{\beta}\frac{d\theta}{d\beta},$$

derive an expression for μ in terms of θ and $\theta_{\rm E}$ only. Discuss the effect of gravitational lensing by a point mass as a point source is moved from (a) $\beta = 0$, to (b) $0 < \beta < \theta_{\rm E}$, and (c) $\beta \gg \theta_{\rm E}$.

(iv) Galaxies are not point-masses. A more realistic model for a lensing galaxy is given by:

$$\kappa(\theta) = \frac{\theta_{\mathrm{E}}}{\theta_{\mathrm{c}}} \left(1 + \frac{\theta^2}{2\theta_{\mathrm{c}}^2}\right) \left(1 + \frac{\theta^2}{\theta_{\mathrm{c}}^2}\right)^{-3/2}$$

where $\kappa(\theta)$ is the convergence inside angular radius θ with mean value:

$$\bar{\kappa}(\theta) \;=\; \frac{\theta_{\mathrm{E}}}{\sqrt{\theta^2 + \theta_{\mathrm{c}}^2}}$$

θ_{c} is the core radius and θ_{E} is a constant.

(a) Deduce $\lim_{\theta\to\infty} \kappa(\theta)$ and $\lim_{\theta\to 0} \kappa(\theta)$. How does $\lim_{\theta\to 0} \kappa(\theta)$ compare with that of a singular isothermal sphere?

(b) What simple condition must θ_{c} satisfy for multiple images to occur?

(c) Find two sets of conditions involving $\kappa(\theta)$ and/or $\bar{\kappa}(\theta)$ which satisfy

$$\det \mathcal{A}(\theta) \;=\; 0\,,$$

where $\mathcal{A}(\theta)$ is the Jacobian matrix of the lens mapping.

(d) Solve one of the sets of conditions for θ in terms of θ_{E} and θ_{c}.

(e) Comment on the significance of the conditions.

4 (i) Explain what is meant by the Cosmological Principle and justify its adoption in modern cosmology. Describe three empirical observations which support it.

(ii) The term 'Consensus Cosmology' refers to a set of cosmological parameters which have now been determined with an accuracy of better than 10%. Among these parameters are:

(a) the Hubble constant $H_0 = 70\,\text{km s}^{-1}\ \text{Mpc}^{-1}$;

(b) the density parameters $\Omega_{b,0} = 0.045$, $\Omega_{m,0} = 0.3$, $\Omega_{\Lambda,0} = 0.7$, $\Omega_{k,0} = 0$, where $\Omega_{\mathrm{i},0}$ indicates the present-day fractional contribution of, respectively, baryons, matter, the cosmological constant, and curvature to the critical density $\rho_{\text{crit}} = 3H_0^2/8\pi G$.

Discuss the observational evidence for these values of the cosmological parameters.

(iii) From the definition of ρ_{crit} at point (ii) and the equation of state of simple fluids:

$$p_i = w_i \rho_i \,,$$

where p_i is the pressure and w_i is a constant for a given component of the universe, give reasons why today's consensus cosmology may appear contrived.

END OF PAPER