UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 1 June, 2006 1.30 to 4.30

PAPER 71

STRUCTURE AND EVOLUTION OF STARS

Attempt ***THREE*** *questions.*
There are ***FOUR*** *questions in total.*
The questions carry equal weight.

The symbols used in these questions have the meanings they were given in the lectures.
Candidates are reminded of the equations of stellar structure in the form:

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \qquad \frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho L_r}{16\pi a c r^2 T^3} \qquad \frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon$$

$$P = \frac{\mathcal{R}\rho T}{\mu} + \frac{aT^4}{3} \qquad \text{with} \quad 1/\mu = 2X + 3Y/4 + Z/2$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A cluster of massive stars is contracting towards the main sequence with the only energy source being due to gravitational contraction. The stellar material is an ideal gas with $\gamma = 5/3$ with radiation pressure being negligible, energy transport is by radiation and the opacity $\kappa = \kappa_0$, where κ_0 is constant.

Show that during this evolution the effective energy production rate per unit mass is given by

$$\epsilon = -\frac{3}{2\rho}\frac{\partial P}{\partial t} + \frac{5P}{2\rho^2}\frac{\partial \rho}{\partial t},$$

with t being the time and the derivative being taken at constant m.

A set of dimensionless variables are defined through $x = r/R$, $q = m/M$, $l = L_r/L$, $b = (4\pi\rho R^3)/M$ and $p = (4\pi R^4 P)/(GM^2)$, with q, l, b, p being functions only of x. The radius R and luminosity L are functions only of time.

Show that in terms of these variables, the equations of stellar structure for a contracting star take the form

$$\frac{dp}{dx} = -\frac{bq}{x^2}, \qquad \frac{dq}{dx} = x^2 b,$$

$$\frac{d}{dx}\left(\frac{p}{b}\right) = -D\frac{b^4 l}{x^2 p^3}, \qquad \frac{dl}{dx} = Ex^2 p,$$

where

$$D = \frac{3\kappa_0 \mathcal{R}^4 L}{64\pi^2 ac\mu^4 G^4 M^3} \quad \text{and}$$

$$E = -\frac{3GM^2}{2R^2 L}\left(\frac{dR}{dt}\right).$$

Hence deduce that the luminosity is $\propto M^3$ and constant during the evolution of a particular star. If the evolution commences at $t = 0$ with very large radius, show that the radius is subsequently given by

$$\frac{RLt}{GM^2} = \text{constant}.$$

The stars eventually reach the main sequence where the energy generation is by the CNO cycle with $\epsilon = \epsilon_0 \rho T^{16}$. Show that they then obey the mass-radius relation

$$R \propto M^{15/19}.$$

Show further that the mass of the stars in the cluster that are just reaching the main sequence at time t satisfies a relation of the form

$$M \propto t^{-19/34}.$$

2 Derive Schwarzschild's condition for stability to convection of a stellar radiative region consisting of an ideal gas with ratio of specific heats $\gamma = 5/3$ in the form

$$\frac{dP}{dr} > \frac{5P}{3\rho}\frac{d\rho}{dr}.$$

Show that this can be written alternatively as

$$\frac{3\kappa L_r P}{16\pi a c G m T^4} < \frac{2}{5}.$$

The temperature in the atmosphere of a cool star is given as a function of the optical depth τ by

$$T^4 = T_e^4\left(\frac{1}{2} + \frac{3}{4}\tau\right)$$

and the opacity is given by $\kappa = \kappa_0 \rho T^{13}$, where κ_0 is constant.

Show that in the upper radiative layers

$$P^2 = \frac{4\pi c G M a \mathcal{R}}{3\kappa_0 L \mu T_e^8}\left(4 - T_e^8/T^8\right)$$

and deduce that convetion sets in when $T = (13/20)^{1/8} T_e$.

In the lower convective region, the structure is polytropic with $P = KT^{5/2}$. Show that if the star is fully convective, there is a relation between the mass, radius and luminosity of the form

$$L \propto M^{8/17} R^{38/17}.$$

3 (a) Show that the electron pressure in a helium gas in which the electrons are completely degenerate but nonrelativistic is given by

$$P = K\rho^{5/3},$$

where

$$K = \left(\frac{3}{2\pi}\right)^{2/3} \frac{h^2}{40 m_e m_p^{5/3}},$$

with h, m_e, and m_p being Planck's constant, the mass of the electron and the mass of a proton respectively.

Deduce that non relativistic helium white dwarfs obey the mass radius relation $R = AM^{-1/3}$, where A is a constant.

(You may assume that for complete degeneracy, the number density of electrons, $n(p)$, with total momentum less than p, is given by $dn(p)/dp = 8\pi p^2/h^3, p < p_0$ and $dn(p)/dp = 0, p > p_0$, where p_o is the Fermi momentum.)

(b) The core of a red giant has mass M_c and is in a regime in which the radius R_c does not vary with M_c.

Above the core is a hydrogen rich radiative envelope which is assumed to have negligible mass. The base of the envelope, at the core surface, coincides with the base of a thin hydrogen burning shell in which the luminosity L is generated. The opacity is given by $\kappa = \kappa_0 \rho / T^{7/2}$ and radiation pressure is neglected.

Assuming the envelope extends to small values of P and T, show that in the regions above the shell

$$P = CT^{17/4},$$

where

$$C = \left(\frac{64\pi acGM_c\mathcal{R}}{51\kappa_0 L\mu}\right)^{1/2}.$$

Show further that T as a function of r is given by

$$T = \frac{4\mu GM_c}{17\mathcal{R}r}.$$

The energy generation rate in the hydrogen rich layers is given by $\epsilon = \epsilon_0 \rho T^{67/4}$. Confirm that the hydrogen burning shell is thin by showing that

$$\epsilon(1.05R_c)/\epsilon(R_c) \sim 1/e$$

and deduce that the luminosity-core mass relation is

$$L \propto M_c^{97/8}.$$

State very briefly what conditions M_c and R_c should satisfy so that this model is consistent.

4 A binary system with components of mass M_1 and M_2 is in circular orbit about the centre of mass with period $P_{orb} = 2\pi/\Omega$. Their distances from the centre of mass are a_1 and a_2 and their angular momenta about the centre of mass are J_1 and J_2 respectively. The separation $a = a_1 + a_2$, and the total orbital angular momentum $J = J_1 + J_2$.

Show that

$$J = \left(\frac{M_1 M_2}{M_1 + M_2}\right) a^2 \Omega = \frac{G^{2/3} P_{orb}^{1/3} M_1 M_2}{(2\pi)^{1/3}(M_1 + M_2)^{1/3}}.$$

The star of mass M_1 is transferring mass to M_2 and simultaneously losing mass to infinity through a stellar wind. The mass transfer rate is $\dot{M}_2 = -f\dot{M}_1$ and the mass loss rate due to the wind is $(1-f)\dot{M}_1$. The wind carries away a specific angular momentum J_1/M_1.

By considering the conservation of angular momentum or otherwise, deduce that

$$P_{orb} \propto M_1^{-3f} M_2^{-3} (M_1 + M_2)^{-2}.$$

The Roche lobe of M_1 is given by the relation

$$R_L = 0.46a \left(\frac{M_1}{(M_1 + M_2)}\right)^{1/3}.$$

Show that

$$\frac{1}{R_L}\frac{dR_L}{dt} = \frac{\dot{M}_1}{M_1}\left(f\left(2q + \frac{4q}{3(1+q)} - 2\right) + \frac{1}{3} - \frac{4q}{3(1+q)}\right),$$

where $q = M_1/M_2$.

The radius R_1 of M_1 is such that $R_1 \propto M_1^{-n}$ and it is assumed to remain in contact with the Roche lobe. Deduce that in that case f must be such that

$$f\left(2q + \frac{4q}{3(1+q)} - 2\right) = \frac{4q}{3(1+q)} - n - \frac{1}{3}.$$

Comment on what happens when q and n are such that the above expression returns a result for f outside the interval $(0, 1)$.

END OF PAPER