UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 8 June, 2006 9 to 12

PAPER 70

ASTROPHYSICAL FLUID DYNAMICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

Candidates are reminded of the equations of ideal magnetohydrodynamics in the form

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \rho = -\rho \boldsymbol{\nabla} \cdot \mathbf{u} \,,$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} p = -\gamma p \boldsymbol{\nabla} \cdot \mathbf{u} \,,$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} \right) = -\rho \boldsymbol{\nabla} \Phi - \boldsymbol{\nabla} p + \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B} \,,$$

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) \,, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0 \,.$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 An unmagnetized polytropic ideal gas of adiabatic exponent $\gamma > 1$ flows in a way such that the fluid variables depend only on x and t, where (x, y, z) are Cartesian coordinates.

(a) Rewrite the governing equations, omitting the gravitational force, to express the conservation of mass, momentum and total energy. Each equation should have the form

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} = 0\,,$$

where Q is the density of the conserved quantity and F is the appropriate flux density.

(b) A stationary shock separates region 1 ($x < 0$) from region 2 ($x > 0$), and is such that $u_x > 0$. Derive the Rankine–Hugoniot relations

$$\begin{aligned}
[\rho u_x]_1^2 &= 0\,,\\
[\rho u_x^2 + p]_1^2 &= 0\,,\\
[\rho u_x u_y]_1^2 &= 0\,,\\
[\rho u_x u_z]_1^2 &= 0\,,\\
[\rho u_x(\tfrac{1}{2}u^2 + w)]_1^2 &= 0\,,
\end{aligned}$$

where

$$w = \left(\frac{\gamma}{\gamma - 1}\right)\frac{p}{\rho}$$

and $[X]_1^2 = X_2 - X_1$ denotes the difference between downstream and upstream values of any quantity X.

(c) Solve the Rankine–Hugoniot relations to show that

$$\frac{u_{x2}}{u_{x1}} = \frac{(\gamma - 1)\mathcal{M}^2 + 2}{(\gamma + 1)\mathcal{M}^2}\,,$$

where $\mathcal{M} = u_{x1}/v_{s1}$ is the shock Mach number and v_s is the adiabatic sound speed. What is the permissible range of $\mathcal{M}$?

(d) Let u_{X2} and u_{Y2} be the downstream velocity components parallel and perpendicular, respectively, to the upstream velocity vector $\mathbf{u}_1$. In the limit of a strong shock, $\mathcal{M} \gg 1$, derive the relation

$$u_{Y2}^2 = (|\mathbf{u}_1| - u_{X2})\left[u_{X2} - \left(\frac{\gamma - 1}{\gamma + 1}\right)|\mathbf{u}_1|\right]\,.$$

Sketch this relation in the (u_{X2}, u_{Y2}) plane. Hence show that the maximum angle through which the velocity vector can be deflected on passing through a stationary strong shock is $\arcsin(1/\gamma)$.

2 (a) Explain why an axisymmetric magnetic field can be represented at any instant of time in the form

$$\mathbf{B} = \boldsymbol{\nabla}\psi(R,z) \times \boldsymbol{\nabla}\phi + B_\phi(R,z)\,\mathbf{e}_\phi\,,$$

where (R, ϕ, z) are cylindrical polar coordinates. Comment on the geometrical and physical significance of the quantity $\psi(R, z)$.

(b) If the magnetic field is also force-free, show that

$$B_\phi = \frac{f(\psi)}{R}\,,$$

where f is an arbitrary function, and that ψ satisfies the equation

$$R^2\boldsymbol{\nabla}\cdot(R^{-2}\boldsymbol{\nabla}\psi) + f\frac{\mathrm{d}f}{\mathrm{d}\psi} = 0\,.$$

(c) Let V be a fixed volume bounded by a surface S. Show that the rate of change of the magnetic energy in V is

$$\frac{1}{\mu_0}\int_S \big[(\mathbf{u}\cdot\mathbf{B})\mathbf{B} - B^2\mathbf{u}\big]\cdot \mathrm{d}\mathbf{S}\,.$$

If V is an axisymmetric volume containing a magnetic field that remains axisymmetric and force-free, and if the velocity on S consists of a differential rotation $\mathbf{u} = R\Omega(R,z)\,\mathbf{e}_\phi$, deduce that the instantaneous rate of change of the magnetic energy in V is

$$\frac{2\pi}{\mu_0}\int f(\psi)\Delta\Omega(\psi)\,\mathrm{d}\psi\,,$$

where $\Delta\Omega(\psi)$ is the difference in angular velocity of the two endpoints on S of the field line labelled by ψ, and the range of integration is such as to cover S once.

3 An unmagnetized polytropic ideal gas of adiabatic exponent $\gamma > 1$ undergoes a steady, spherically symmetric accretion flow on to a point mass M. At infinity, the gas is at rest and has uniform density ρ_0 and adiabatic sound speed v_{s0}.

(a) Show that the flow can pass smoothly through a sonic point only if $\gamma < 5/3$.

(b) If $\gamma = 5/3$, show that the Mach number $\mathcal{M} = |u_r|/v_s$ is related to the radius r by

$$\rho_0^{-1/2} v_{s0}^{3/2} \left(\frac{\dot{M}}{4\pi}\right)^{1/2} \left(\frac{1}{2}\mathcal{M}^{3/2} + \frac{3}{2}\mathcal{M}^{-1/2}\right) = \frac{3}{2} v_{s0}^2 r + GM\,,$$

where $\dot{M}$ is the mass accretion rate. Sketch the possible solution curves in the $(r, \mathcal{M})$ plane for various values of $\dot{M}$. Comment on which solutions are physically acceptable.

(c) Show that the maximum possible accretion rate in the case $\gamma = 5/3$ is $\pi G^2 M^2 \rho_0 v_{s0}^{-3}$.

4 A polytropic ideal gas of adiabatic exponent γ forms a static atmosphere in a constant gravitational field $-g\,\mathbf{e}_z$, where (x, y, z) are Cartesian coordinates. A horizontal magnetic field $B(z)\,\mathbf{e}_y$ is also present.

(a) Derive the linearized equations

$$\delta\rho = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\rho - \rho\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\,,$$
$$\delta p = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p - \gamma p\boldsymbol{\nabla}\cdot\boldsymbol{\xi}\,,$$
$$\delta\mathbf{B} = -\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\mathbf{B} + \mathbf{B}\cdot\boldsymbol{\nabla}\boldsymbol{\xi} - \mathbf{B}(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})\,,$$
$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} = -\boldsymbol{\nabla}\delta\Pi - g\,\delta\rho\,\mathbf{e}_z + \frac{1}{\mu_0}(\delta\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{B} + \mathbf{B}\cdot\boldsymbol{\nabla}\delta\mathbf{B})\,,$$
$$\delta\Pi = \delta p + \frac{\mathbf{B}\cdot\delta\mathbf{B}}{\mu_0}$$

governing small displacements $\boldsymbol{\xi}$.

(b) For a displacement that is periodic in x and y and is of the form

$$\mathrm{Re}\left[\boldsymbol{\xi}(z)\exp(-\mathrm{i}\omega t + \mathrm{i}k_x x + \mathrm{i}k_y y)\right]\,,$$

show that

$$\omega^2\int_a^b \rho|\boldsymbol{\xi}|^2\,\mathrm{d}z = [\xi_z^*\delta\Pi]_a^b$$
$$+\int_a^b\left(\frac{|\delta\Pi|^2}{\gamma p + \frac{B^2}{\mu_0}} - \frac{\left|\rho g\xi_z + \frac{B^2}{\mu_0}\mathrm{i}k_y\xi_y\right|^2}{\gamma p + \frac{B^2}{\mu_0}} + \frac{B^2}{\mu_0}k_y^2|\boldsymbol{\xi}|^2 - g\frac{\mathrm{d}\rho}{\mathrm{d}z}|\xi_z|^2\right)\mathrm{d}z\,, \qquad (*)$$

where $z = a$ and $z = b$ are the lower and upper boundaries of the atmosphere.

(c) You may assume that the atmosphere is unstable if and only if the integral on the right-hand side of equation $(*)$ can be made negative by a trial displacement $\boldsymbol{\xi}$ satisfying the boundary conditions, which are such that $[\xi_z^*\delta\Pi]_a^b = 0$. Explain why the integral can be minimized with respect to ξ_x by letting $\xi_x \to 0$ and $k_x \to \infty$ in such a way that $\delta\Pi = 0$.

(d) Hence show that the atmosphere is unstable to disturbances with $k_y = 0$ if and only if

$$-\frac{\mathrm{d}\ln\rho}{\mathrm{d}z} < \frac{\rho g}{\gamma p + \frac{B^2}{\mu_0}}$$

at some point.

(e) Assuming that the condition in part (d) is not satisfied anywhere, show also that the atmosphere is unstable to disturbances with $k_y \neq 0$ if and only if

$$-\frac{\mathrm{d}\ln\rho}{\mathrm{d}z} < \frac{\rho g}{\gamma p}$$

at some point.

END OF PAPER