UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Friday 2 June, 2006 1.30 to 4.30

## PAPER 7

## TOPICS IN FOURIER ANALYSIS

*Attempt* ***THREE*** *questions.*

*There are* ***FOUR*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

(i) Show that, if $f : \mathbb{T} \to \mathbb{C}$ is continuous, then

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|} e^{int} = f * P_r(t)$$

where

$$P_r(t) = \frac{1 - r^2}{1 - 2r\cos t + r^2}.$$

By using the proof of Fejér's theorem as a model, or otherwise, show that

$$f * P_r(t) \to f(t)$$

uniformly as $r \to 1-$.

(ii) If $p$ is a prime of the form $4k+1$, then it can be shown that there exists an integer $u$ with $u^2 \equiv -1 \bmod p$. By considering the lattice

$$\Lambda = \{(n, m) \in \mathbb{Z}^2 \;:\; m \equiv nu \bmod p\},$$

show, carefully stating any theorems that you use, that $p$ is the sum of two squares.

**2** Prove the theorem of Kahane and Katznelson which states that, given any subset $E$ of $\mathbb{T}$ of Lebesgue measure zero, there exists an $f \in C(\mathbb{T})$ whose Fourier sum diverges on $E$.

**3**

(i) Show that, if $f : \mathbb{R} \to \mathbb{C}$ is reasonably behaved, then

$$\frac{\int_{-\infty}^{\infty} x^2 |f(x)|^2 \, dx}{\int_{-\infty}^{\infty} |f(x)|^2 \, dx} \times \frac{\int_{-\infty}^{\infty} \lambda^2 |\hat{f}(\lambda)|^2 \, d\lambda}{\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 \, d\lambda} \geqslant \frac{1}{4}.$$

You should quote carefully any theorems that you use. Explain, with proof, when equality occurs.

(ii) Show how, if $f : \mathbb{R} \to \mathbb{C}$ is reasonably behaved and $\hat{f}(\lambda) = 0$ for $|\lambda| \geqslant \pi$, we can recover $f(t)$ for all $t \in \mathbb{R}$ from the values of $f(n)$ with $n$ an integer.

Show that, if $\epsilon > 0$, the condition $\hat{g}(\lambda) = 0$ for $|\lambda| \geqslant \pi + \epsilon$, does not imply that $g$ is uniquely determined by the values of $g(n)$ with $n$ an integer.

**4**

(i) Prove the following version of the Riemann–Lebesgue lemma. If $f : \mathbb{T} \to \mathbb{C}$ is continuous, then $\hat{f}(r) \to 0$ as $|r| \to \infty$.

Show also that, if $k(r) > 0$ and $k(r) \to \infty$ as $r \to \infty$, then there exists a continuous function $g : \mathbb{T} \to \mathbb{C}$ such that $\limsup_{r\to\infty} k(r)|\hat{g}(r)| = \infty$.

(ii) In this question we use the normalisation of Haar functions such that

$$\int |H(t)|^2 \, dt = 1$$

for each Haar function $H$. Prove the case of Tao's theorem which states that, if $f : \mathbb{T} \to \mathbb{C}$ is continuous, then

$$\sum_{|\hat{f}(H)|\geqslant\delta} \hat{f}(H)H(t) \to f(t)$$

uniformly as $\delta \to 0+$.

## END OF PAPER