UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 7 June, 2006 1.30 to 4.30

PAPER 68

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Attempt ***THREE*** *questions from Section A and* ***ONE*** *question from Section B.*

Each question from Section B carries twice the weight of a question from Section A.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 **(a)** Prove that any ν-stage Runge–Kutta method of order 2ν is necessarily A-stable. [*You may use, without proof, properties of Padé approximants to the exponential.*]

(b) Describe (without proof) how to construct explicitly the coefficients of such a method.

2 The equation

$$\frac{\partial}{\partial t}u = \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}u, \qquad t \geqslant 0, \quad 0 \leqslant x, y \leqslant 1,$$

with periodic boundary conditions, is approximated by the fully discretized scheme

$$u_{k,j}^{n+1} = \mu(u_{k+1,j}^n + u_{k,j+1}^n - u_{k-1,j}^n - u_{k,j-1}^n) + u_{k,j}^{n-1},$$

where $u_{k,j}^n \approx u(k\Delta x, j\Delta x, n\Delta t)$ and $\mu = \Delta t/\Delta x$.

(a) Determine the order of the method.

(b) Find the range of $\mu > 0$ that yields stability.

3 Consider the multistep method with the polynomials

$$\rho(w) = w^3 - (1+2\alpha)w^2 + (1+2\alpha)w - 1, \qquad \sigma(w) = \frac{1}{6}[(5+\alpha)w^3 - (4+8\alpha)w^2 + (11-5\alpha)w].$$

(a) For which values of α is the method convergent?

(b) What is the order of the method for different values of α?

(c) For which values of α is the method A-stable?

4 We consider the two-point boundary-value problem for the Airy equation

$$u'' - xu = 0, \qquad u(0) = 1, \quad u'(1) = 0.$$

(a) Show that the problem can be written in the form $\mathcal{L}(u) = 0$, where $\mathcal{L}$ is a positive definite operator. Thereby, quoting appropriate definitions and theorems, formulate a variational problem whose unique minimum is the solution of the equation.

(b) The above variational problem is approximated with the Ritz method, using hat functions. Derive explicitly the discretized equations.

SECTION A (continued)

5 The advection equation $\partial u/\partial t = \partial u/\partial x$ is solved by the two-step finite difference method

$$u_m^{n+1} = a_{-2}(\mu)u_{m-2}^n + a_{-1}(\mu)u_{m-1}^n + a_1(\mu)u_{m+1}^n + a_2(\mu)u_{m+2}^n + u_m^{n-1},$$

where $\mu = \Delta t/\Delta x$.

(a) Find functions $a_k(\mu)$, $k = \pm 1, \pm 2$, such that the method is of order at least four.

(b) Assuming that we are solving the Cauchy problem, prove that $\mu = \frac{1}{2}$ gives a stable method, while the choice $\mu = \frac{3}{2}$ results in instability.

SECTION B

6 Describe the Engquist–Osher method for a single, one dimensional, hyperbolic nonlinear conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0.$$

Prove that it is stable, provided that f is convex, differentiable and possesses a unique stagnation (sonic) point.

7 Write an essay on the *Mehrstellenverfahren* (finite difference methods of added accuracy) for the Poisson equation.

END OF PAPER