UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 9 June, 2006 9 to 12

PAPER 63

BLACK HOLES

Attempt ***THREE*** *questions.*

There are ***FOUR*** *question on this paper.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 What is meant by the concept of a vierbein field in general relativity?

In what sense is general relativity invariant under Lorentz transformations?

How does one define a spinor field?

The covariant derivative of e_a^μ, the vierbein, (a is a curved spacetime index and μ is a flat Lorentz index) vanishes:

$$\nabla_a e_b^\mu = 0.$$

Use this to derive a formula for the spin connection in terms of the Christoffel symbol Γ^a_{bc}, the vierbein and its derivatives.

Suppose the spacetime is conformally flat, so that the metric can be written

$$g_{ab} = \Omega^2 \eta_{ab},$$

where Ω is a smooth positive function and η_{ab} is the flat metric on Minkowski space in Minkowskian coordinates (t, x, y, z). Evaluate the spin connection.

Hence show, given a flat spacetime constant spinor ϵ_0 that obeys $\partial_a \epsilon_0 = 0$ when written in Minkowskian coordinates (t, x, y, z), that $\epsilon = \sqrt{\Omega}\epsilon_0$ satisfies the equation

$$\left[\gamma_a \nabla_b + \gamma_b \nabla_a - \frac{1}{2} g_{ab} \gamma^c \nabla_c\right] \epsilon = 0,$$

where ∇_a is the covariant derivative operator for spinors in the metric g_{ab}.

2 In five-dimensional spacetime, the analogue of the Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2m}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2, \qquad (*)$$

where $m > 0$ is the mass parameter and $d\Omega_3^2$ is the line element on a unit three-sphere that has proper area $2\pi^2$.

Construct the analogue of Eddington-Finkelstein coordinates for this metric.

Find the analogues of the Eddington-Finkelstein forms of the metric in these coordinates.

Use your results to construct the Penrose diagram for this spacetime.

The mass of the black hole can be evaluated from the Komar integral

$$\int_{\iota^0} \nabla_a k_b \; d\Sigma^{ab},$$

where k^a is the Killing vector associated with time translations. Briefly describe why this integral should be proportional to the mass. Evaluate this integral for the metric $(*)$.

3 Sketch the derivation of the area theorem for black holes.

One suggested modification of general relativity involves the introduction of a cosmological constant Λ, so that the Einstein equations become

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab}.$$

If the energy-momentum tensor of matter obeys the condition $T_{ab} p^a p^b \geq 0$ for arbitrary future-directed null vectors p^a, explain whether or not you would expect the area theorem to hold.

A different modification of general relativity is to add matter that involves a massless scalar field ϕ that makes a contribution to the energy-momentum tensor of

$$T_{ab} = (\partial_a \phi)(\partial_b \phi) - \frac{1}{2} g_{ab} (\partial_c \phi)(\partial_d \phi) g^{cd}.$$

Carefully explain whether you would expect the area theorem to hold in this case.

4 A scalar field ϕ is quantised in a background spacetime metric g_{ab}. In the remote past and the distant future, this metric is time independent. Describe in detail how to use the idea of Bogoliubov transformations to describe how the state of ϕ changes between the remote past and the distant future.

Suppose that in the remote past the system is in the vacuum state $|0_-\rangle$. Find in terms of the Bogoliubov coefficients an expression for the expected number of particles $\langle s|N_s|s\rangle$ created in the state $|s\rangle$ as seen in the distant future. Similarly find an expression for $\langle s|N_s^2|s\rangle$.

Sketch how considerations of this type lead to a picture of particle production by black holes.

END OF PAPER