UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 5 June, 2006 1.30 to 4.30

PAPER 62

COSMOLOGY

*Attempt **ALL** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Starting from the action

$$\mathcal{S} = \frac{m}{2}\int d\lambda \left(\frac{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}{e} - e \right),$$

for a particle of mass m in a background metric $g_{\mu\nu}$:

(a) Derive the geodesic equation and the constraint equation (mass shell condition). Explain why it is permissible to set $e = 1$ in these equations.

(b) Taking the background line element to be $-dt^2 + a^2(t)d\mathbf{x}^2$, find the canonical spatial momentum $\mathbf{p}^C$ conjugate to $\mathbf{x}$ and show that it is conserved. Defining the kinetic momentum as $\mathbf{p}^K = \gamma m \mathbf{v}$ where $\mathbf{v} = a d\mathbf{x}/dt$ is the (proper) 3-velocity of the particle and $\gamma = (1 - v^2)^{-\frac{1}{2}}$, show that $\mathbf{p}^K = \mathbf{p}^C/a$.

(c) Express $d\mathbf{x}/dt$ in terms of $\mathbf{p}^C$. If the particle has a kinetic momentum p_0 at t_0, show that the total comoving distance the particle has moved since the big bang, taken to be at $t = 0$, is

$$d = \int_0^{t_0} \frac{dt}{a(t)} \frac{1}{\sqrt{1 + (ma(t)/p_0)^2}},$$

where $a(t_0) = 1$.

(d) Consider a radiation dominated universe, and a particle whose kinetic momentum p equals the radiation temperature T_r, at all times. Show that the total comoving distance the particle has moved is given, at any time, by

$$d = H^{-1}\frac{T_r}{m}\sinh^{-1}(\frac{m}{T_r}),$$

where H^{-1} is the Hubble radius at that time. Use this to estimate d/H^{-1} for a neutrino of mass 30 eV, at the end of the radiation era. (Today's CMB temperature is approximately 2.35 meV and you may assume the redshift of equal matter/radiation density is $2.5 \times 10^4 \Omega_M h^2$). What comoving length scale does this 'neutrino streaming length' translate into in today's universe?

2 The action for a scalar field ϕ with potential $V(\phi)$ is

$$\mathcal{S} = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right).$$

(a) Show that in a flat FRW universe with line element $-dt^2 + a^2(t)d\mathbf{x}^2$, the equation of motion for a spatially homogeneous scalar field $\phi(t)$ is

$$\ddot{\phi} + 3H\dot{\phi} = -\frac{\partial V}{\partial\phi}, \qquad (*)$$

where dots denote time derivatives and $H = \dot{a}/a$.

(b) Calculate the stress energy tensor $T^{\mu\nu} = \frac{2}{\sqrt{-g}}(\delta S/\delta g_{\mu\nu})$, for the scalar field. Using the relation $T^\mu_\nu = \mathrm{diag}(-\rho, P, P, P)$ for a homogeneous, isotropic fluid, show that for a spatially homogeneous scalar field, $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and $P = \frac{1}{2}\dot{\phi}^2 - V(\phi)$.

(c) Write down the Friedmann equation for gravity coupled to such a scalar field. Using this, and equation $(*)$, derive the deceleration equation (i.e. the equation expressing $\ddot{a}/a$ in terms of the scalar field and its time derivative). Show that the condition for inflation, *i.e.*, $\ddot{a} > 0$, is $V(\phi) > \dot{\phi}^2$.

(d) Show that if we assume that the scalar field starts with negligible kinetic energy, the condition for inflation to last at least of order one Hubble time H^{-1} is $M_{Pl}^2 V_{,\phi}^2/V^2 < 1$, where $M_{Pl} = (8\pi G)^{-\frac{1}{2}}$. Show also that if, in addition, the acceleration term in equation $(*)$ is to remain negligible, we must also have $M_{Pl}^2 V_{,\phi\phi}/V \ll 1$.

3 In thermal equilibrium at a temperature T, the number density of a particle species i with mass $m_i \gg T$ and chemical potential μ_i is

$$n_i = g_i \left(\frac{m_i T}{2\pi}\right)^{\frac{3}{2}} \exp\left(\frac{\mu_i - m_i}{T}\right).$$

(a) From this formula show that the relative abundance of Hydrogen atoms, protons and electrons are related, to good accuracy, by

$$\frac{n_H}{n_p n_e} = \left(\frac{2\pi}{m_e T}\right)^{\frac{3}{2}} \exp(I/T),$$

where I=13.6 eV is the ionisation energy of Hydrogen.

(b) Hence derive Saha's equation for the ionisation fraction $X_e = n_e/n_B$, ignoring Helium:

$$\frac{1 - X_e}{X_e^2} = \frac{2\zeta(3)}{\pi^2}\eta \left(\frac{2\pi T}{m_e}\right)^{\frac{3}{2}} \exp(I/T),$$

where $\eta = n_B/n_\gamma \approx 2.68 \times 10^{-8}\Omega_B h^2$ is the Baryon-to-photon ratio and you may assume $n_\gamma = (2\zeta(3)/\pi^2)T^3$.

(c) Explain why the ionisation fraction falls below a few per cent when the temperature of the universe falls below ~ 0.3 eV, not 13.6 eV as one might naively expect.

(d) Assume that the universe is flat and matter dominated today, and that the CMB temperature is approximately 2.35 meV. Estimate the angle subtended by the Hubble horizon when protons and electrons combine, as seen by an observer looking out to the CMB sky today.

4 Cosmological perturbation theory is traditionally described in synchronous gauge, $\delta g_{0\mu} = 0$.

(a) Explain why it is always possible to choose this gauge.

(b) Consider cold dark matter, with background density ρ_C and with perturbed density $\rho_C(1+\delta_C)$. To first order in the perturbations, the equation for stress energy conservation, $T^{\mu\nu}{}_{;\nu} = 0$ yields

$$\dot{\delta_C} + \nabla \cdot \mathbf{v_C} + \frac{1}{2}\dot{h} = 0, \qquad \dot{\mathbf{v_C}} + \frac{\dot{a}}{a}\mathbf{v_C} = 0.$$

Here, the perturbed metric has been taken in conformal time and synchronous gauge so the line element is $a^2(\tau)\left(-d\tau^2 + dx^i dx^j(\delta_{ij} + h_{ij}(\tau, \mathbf{x}))\right)$, and $h \equiv \mathrm{Tr}(h_{ij})$. Dots denote conformal time derivatives and $\mathbf{v_C}$ is the CDM velocity in the $(\tau, \mathbf{x})$ coordinates. Explain why, even if $\mathbf{v_C}$ is significant initially, it decays away so that it is negligible at late times.

(c) The linearised Einstein equations yield the evolution equation for the trace of the metric perturbation,

$$\ddot{h} + \frac{\dot{a}}{a}\dot{h} = -8\pi G a^2 \sum_N (1 + 3c_N^2)\rho_N \delta_N.$$

Consider a cold dark matter dominated universe. Assuming $\mathbf{v_C}$ to be negligible, show that the general solution for δ_C is

$$\delta_C = A(\mathbf{x})\tau^2 + B(\mathbf{x})\tau^{-3}.$$

(d) Now, consider a radiation dominated universe which contains a sub-dominant component of cold dark matter. It is a reasonable approximation to treat the radiation as being uniform on scales much smaller than the Hubble radius. Show that, with this approximation, the general solution for δ_C is

$$\delta_C = A(\mathbf{x})\ln(\tau) + B(\mathbf{x}).$$

Explain why the logarithmic term might be significant for the growth of galaxy-scale perturbations in a realistic cosmology.

END OF PAPER