UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Friday 2 June, 2006   9 to 12

PAPER 61

GENERAL RELATIVITY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

The signature is $(+ - - -)$ *and the curvature tensor conventions are defined by*

$$R^i{}_{kmn} = \Gamma^i{}_{km,n} - \Gamma^i{}_{kn,m} - \Gamma^i{}_{pm}\Gamma^p{}_{kn} + \Gamma^i{}_{pn}\Gamma^p{}_{km} \, .$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Spacetime has a metric g_{ik}, we are using the metric connection and the torsion tensor vanishes. Establish the identities

$$R^i{}_{k(mn)} = 0, \qquad R^i{}_{[kmn]} = 0, \qquad R_{(ik)mn} = 0, \qquad R_{ikmn} = R_{mnik},$$

the Bianchi identities

$$R^i{}_{k[mn;p]} = 0,$$

and their contracted form. Obtain also the Ricci identity for vector fields X^i

$$X^i{}_{;km} - X^i{}_{;mk} = R^i{}_{jkm} X^j,$$

and the equivalent identity for covector fields ω_i.

Show also that

$$R^{ik}{}_{mn;ik} = 0.$$

2 One attempt at making Newtonian gravitation theory covariant is to rewrite Newton's second law in differential form as

$$dp^i = \eta^{ik} \Phi_{,k} p_j dx^j - p^j \Phi_{,j} dx^i,$$

where p^i is the 4-momentum of a particle (whose mass is defined via $m^2 = p^i p_i$), η^{ik} is the Minkowski metric and the scalar field Φ is related to the energy-momentum tensor via

$$\Phi^{;i}{}_i = -4\pi G T^i{}_i.$$

Answer the following questions giving reasons for your answers.

(a) Is this theory in agreement with the experiments of Eötvös?

(b) Is this theory in agreement with the Pound-Rebka experiment?

(c) Is this theory consistent with the observed bending of light by the Sun?

3 For a linearized perturbation of Minkowski spacetime with metric

$$g_{ik} = \eta_{ik} + \epsilon h_{ik}$$

where $\epsilon \ll 1$, show that the Riemann curvature tensor is

$$R_{ikmn} = \tfrac{1}{2}\epsilon(h_{im,kn} + h_{nk,im} - h_{mk,in} - h_{in,km}).$$

Explain the concept of *gauge* in this context and show, explicitly, that R_{ikmn} is gauge-invariant.

Show that, introducing an auxiliary tensor $\bar{h}_{ik}$ and a particular choice of gauge (both to be specified), the linearized form of the Einstein field equations $R_{ik} - \frac{1}{2}g_{ik}R = -8\pi G T_{ik}$ becomes

$$\Box \bar{h}_{ik} = -16\pi G \epsilon^{-1} T_{ik},$$

where the operator $\Box$ should be defined.

A weak beam of photons is moving in the x-direction (we are using cartesian coordinates) and the only non-vanishing components of the energy-momentum tensor are $T_{00} = T_{11} = -T_{01} = -T_{10} = \epsilon\rho(t - x)$. What can be said about the components of h_{ik}?

Deduce that other photons moving parallel to the beam experience no deflection.

4 Toy (1 + 1-dimensional) models with line elements for *de Sitter spacetime*

$$ds^2 = dt^2 - \cosh^2 t \, d\chi^2,$$

and *anti-de Sitter spacetime*

$$ds^2 = \cosh^2 r \, dt^2 - dr^2,$$

were introduced in the lectures. What are the ranges of the various coordinates?

Write an essay on the similarities and differences between these two models, paying particular attention to the geodesics, conformal structure and the horizons.

END OF PAPER