UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 13 June, 2006 9 to 11

PAPER 60

CONTROL OF QUANTUM SYSTEMS

*Attempt **FOUR** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Basic concepts. Consider the following four density operators for a system with Hilbert space dimension $N = 2$ (qubit):

$$\hat{\rho}_0 = \frac{1}{4}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\rho}_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \hat{\rho}_2 = \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad \hat{\rho}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(a) Compute the eigenvalues for the density matrices above.

(b) Which of the density matrices represent pure states? Give an equivalent wavefunction representation.

(c) Which of the states are unitarily equivalent?

(d) Assume the two-level system evolves according to the dynamical law (setting $\hbar = 1$)

$$i\tfrac{d}{dt}\hat{\rho}(t) = \left[\hat{H}[f(t)], \hat{\rho}(t)\right],$$

where $\hat{H}[f(t)] = \hat{\sigma}_z + f(t)\hat{\sigma}_x$, $[A, B] = AB - BA$ is the usual matrix commutator and the Pauli matrices are:

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Which of the states $\hat{\rho}_k$, $k = 0, 1, 2, 3$, above can be inter-converted by applying a suitable (open-loop) control function? Briefly justify your answer.

NB: You do *not* need to find a suitable control. A simple existence argument based on the controllability of the system and kinematical constraints will suffice.

2 Criteria for Controllability of Hamiltonian Systems. The following statements relate to (Hamiltonian) quantum control systems governed by a control Hamiltonian

$$\hat{H}[\mathbf{f}(t)] = \hat{H}_0 + \sum_{m=1}^{M} f_m(t)\hat{H}_m, \tag{1}$$

where the $f_m(t)$ are independent control fields and $\hat{H}_m$ (for $m = 0, 1, \ldots, M$) are Hermitian operators. An N-level system is a system with Hilbert space dimension N. We shall assume that M, N are integers with $1 \leq M < \infty$, $1 < N < \infty$.

For each of the statements below, decide if it is true or false and briefly justify your answer.

(a) If the dynamical Lie algebra generated by an N-level system with control Hamiltonian (1) has dimension $N^2 - 1$ then

a1: the system is controllable.

a2: we can dynamically generate any unitary operator with determinant one, or equivalently, any unitary operator up to a global phase.

a3: we can dynamically generate *any* unitary operator.

(b) Consider a system of (even) dimension $N = 2\ell$. The anti-Hermitian matrices $i\hat{H}_m$ (for $m = 0, 1, \ldots, M$) satisfy the relation $\hat{x}^T\hat{J} + \hat{J}\hat{x} = 0$ for $\hat{x} = i\hat{H}_m$ with $\hat{J} = \begin{pmatrix} 0 & -\hat{I}_\ell \\ \hat{I}_\ell & 0 \end{pmatrix}$, where $\hat{I}_\ell$ is the identity matrix in dimension ℓ. Then

b1: the system is controllable.

b2: the system is pure-state controllable.

If there is insufficient information, what additional information would be necessary to decide?

(c) Pure-state controllability implies controllability if

c1: the system dimension is $N = 2$.

c2: the system dimension is even and greater than 2.

(d) A Morse oscillator with non-zero anharmonicity ($B \neq 0$) is controllable if all transitions between adjacent elements (and only those) are allowed.

3 Consider a control-linear system governed by the control Hamiltonian

$$\hat{H}[\mathbf{f}(t)] = \hat{H}_0 + \sum_{m=1}^{M} f_m(t)\hat{H}_m,$$

where the $f_m(t)$ are independent control fields and the $\hat{H}_m$, $m = 0, 1, \ldots, M$ are Hermitian operators. An N-level system is a system with Hilbert space dimension N. We shall assume that M, N are integers with $1 \leq M < \infty$, $1 < N < \infty$.

(a) Let $\hat{U}(t, t_0)$ be the time-evolution operator of the system satisfying the Schrödinger equation (setting $\hbar = 1$)

$$i\tfrac{d}{dt}\hat{U}(t, t_0) = \hat{H}[\mathbf{f}(t)]\hat{U}(t, t_0),$$

and set $\hat{U}(t, t_0) = \hat{U}_0(t, t_0)\hat{U}_I(t, t_0)$ where $\hat{U}_0 = \exp[-it\hat{H}_0]$. Show that $\hat{U}_I(t, t_0)$ satisfies

$$i\frac{d}{dt}\hat{U}_I(t, t_0) = \sum_{m=1}^{M} f_m(t)\tilde{H}_m(t)\hat{U}_I(t, t_0),$$

with $\tilde{H}_m(t) = \hat{U}_0(t, t_0)^\dagger \hat{H}_m \hat{U}_0(t, t_0)$.

(b) Consider a control-linear system as above with $N = 2$, $M = 1$ and $\hat{H}_0 = (\omega/2)\hat{\sigma}_z$, $\hat{H}_1 = \hat{\sigma}_x$ where $\hat{\sigma}_x$ and $\hat{\sigma}_z$ are the usual Pauli matrices. Show that if we apply a field of the form $f(t) = A(t)\cos(\omega t + \phi)$ the rotating wave approximation Hamiltonian is

$$f(t)\tilde{H}_1(t) = f(t)\hat{U}_0(t, t_0)^\dagger \hat{H}_1 \hat{U}_0(t, t_0) \approx \frac{A(t)}{2}\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}.$$

[*You may use the following result:*

$$\hat{U}_0(t, t_0)^\dagger \hat{\sigma}_x \hat{U}_0(t, t_0) = \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}.$$]

(c) Consider a (strongly regular) 4-level system with allowed transitions as shown in Figure 1 below. Assume the system is initially in state $|1\rangle$. Describe *briefly* how we can, in principle, transfer the system to state $|4\rangle$ by applying a sequence of simple control pulses of the form $f_k(t) = A_k(t)\cos(\mu_k t + \phi_k)$ where $\mu_k \in \{\omega_1, \omega_2, \omega_3\}$.

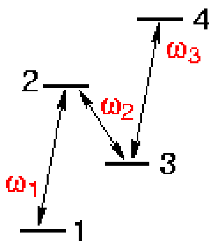

Fig 1: Level diagram ($\omega_1 \neq \omega_2 \neq \omega_3 \neq \omega_1$)

(d) Suppose we would like to instead prepare the system in the superposition state $|\Psi\rangle = [|1\rangle - |2\rangle + |4\rangle]/\sqrt{3}$. *Outline* how we could solve this problem using geometric control.

4 Optimal Control. Consider the variational optimal control functional $\mathcal{J} = \mathcal{A} - \mathcal{D} - \mathcal{C}$, where

$$\mathcal{A}[\rho_v] = \langle\langle \hat{A} \mid \rho_v(t_F)\rangle\rangle,$$
$$\mathcal{D}[\mathbf{f}, \rho_v, A_v] = \int_{t_0}^{t_F} \langle\langle A_v(t)| \frac{\partial}{\partial t} + \frac{i}{\hbar}\mathcal{L}_{tot}[\mathbf{f}(t)]|\rho_v(t)\rangle\rangle \, dt,$$
$$\mathcal{C}[\mathbf{f}] = \frac{p_0}{\hbar} \sum_{m=1}^{M} \frac{\lambda_m}{2} \int_{t_0}^{t_F} f_m^2(t) \, dt,$$

with $\mathcal{L}_{\text{tot}}[\mathbf{f}(t)] = \mathcal{L}_0 + \mathcal{L}_D + \sum_{m=1}^{M} f_m(t)\mathcal{L}_m$ and $\mathbf{f}(t) = (f_1(t), \ldots, f_M(t))$.

(a) For the functional $\mathcal{J}$, identify the objective, dynamic constraints and penalty terms. What do the operators ρ_v and A_v represent? What is the significance of the parameters λ_m?

(b) Give necessary conditions for $\mathcal{J}$ to have an extremum, and state the Euler-Lagrange equations for the functional above. (You do *not* need to derive the Euler-Lagrange equations.)

(c) Write down a possible objective functional for each of the following control problems. (Assume an N-level system, $N < \infty$, and $t_0 = 0$.)

c1: Maximize the population of state $|5\rangle$ at time $t_F = 10$.

c2: Maximize the expectation value of the projection onto the state $|\Psi\rangle = (|1\rangle - i|5\rangle)/\sqrt{2}$ at time $t_F = 100$.

c3: Maximize the energy of the system at time t_F.

(d) Describe briefly how you could implement a complicated shaped (optical) pulse such as the one shown in Fig. 2 *experimentally* using pulse shaping equipment.

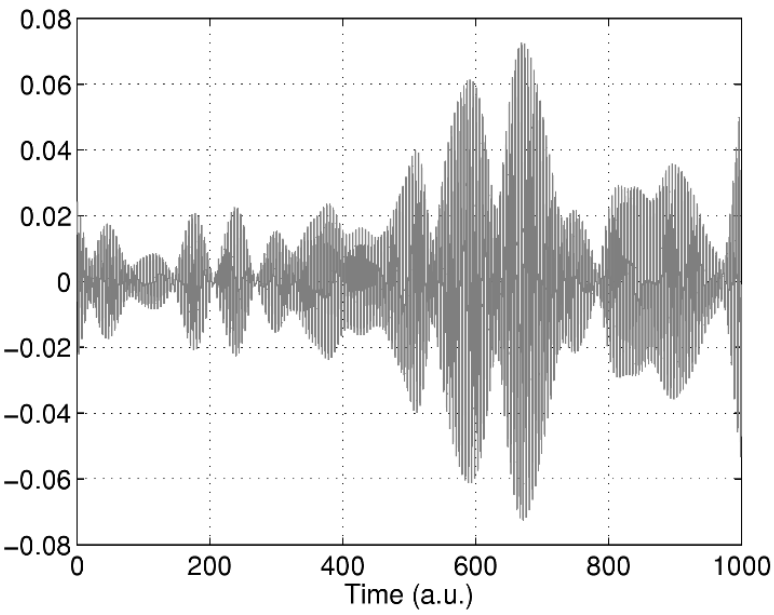

Fig 2: Optimally Shaped Pulse

5 Adiabatic Control. Consider a three-level system in a Λ configuration as in Figure 3. Two lasers are available to *resonantly* excite the $|1\rangle \leftrightarrow |2\rangle$ and $|2\rangle \leftrightarrow |3\rangle$ transition, respectively. Assuming the fields are of the form $f_{12}(t) = A_{12}(t)\cos(\omega_{12}t)$ and $f_{23}(t) = A_{23}(t)\cos(\omega_{23}t)$, where $A_{12}(t)$ and $A_{23}(t)$ are slowly varying envelope functions, the interaction picture Rotating Wave Approximation Hamiltonian for the system is

$$\hat{H}^{\mathrm{RWA}}[\Omega_{12}(t), \Omega_{23}(t)] = \begin{pmatrix} 0 & \Omega_{12}(t) & 0 \\ \Omega_{12}(t) & 0 & \Omega_{23}(t) \\ 0 & \Omega_{23}(t) & 0 \end{pmatrix},$$

where $\Omega_{12}(t) = A_{12}(t)d_{12}/2\hbar$ and $\Omega_{23}(t) = A_{23}(t)d_{23}/2\hbar$ are the (effective) pulse envelopes of laser fields $f_{12}(t)$ and $f_{23}(t)$, respectively, and d_{12} and d_{23} are the respective dipole moments.

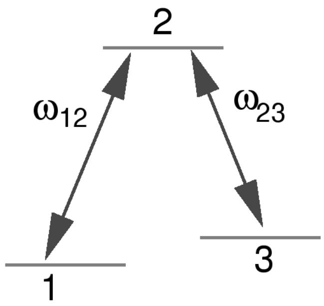

Fig 3: 3-level Λ system

(a) Show that $|\Psi_0(t)\rangle = \cos\theta(t)|3\rangle - \sin\theta(t)|1\rangle$, where $\theta(t) = \arctan[\Omega_{23}(t)/\Omega_{12}(t)]$, is an eigenstate of the RWA Hamiltonian with eigenvalue $\lambda = 0$.

[*You may find the following trigonometric identities useful:*

$$\cos(x) = \frac{1}{\sqrt{1+\tan^2(x)}}, \quad \sin(x) = \frac{\tan(x)}{\sqrt{1+\tan^2(x)}}.$$]

(b) Assume the system is initially in the state $|3\rangle$. Describe how we can *adiabatically* transfer the population from the state $|3\rangle$ to the state $|1\rangle$. Provide a sketch of the pulse sequence.

(c) Is this scheme robust with regard to (i) decay of the excited state $|2\rangle$, (ii) perturbations of the pulse envelopes, and (iii) decoherence of $|1\rangle \leftrightarrow |3\rangle$ transition? Justify your answers briefly.

(d) When during the pulse sequence is the eigenstate $|\Psi_0(t)\rangle$, as defined in (a), equal to $(|3\rangle - |1\rangle)/\sqrt{2}$?

END OF PAPER