UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 2 June, 2006 9 to 12

PAPER 6

INTRODUCTION TO BANACH SPACES AND ALGEBRAS

Attempt ***FOUR*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let E be a Banach space, F a normed space and let T be a bounded linear mapping from E to F. Suppose that there exist constants $R > 0$ and $0 < k < 1$ such that, for every $y \in F$ with $\|y\| \leqslant 1$, there is some $x \in E$ with $\|x\| \leqslant R$ such that $\|Tx - y\| \leqslant k$. Prove that (i) for every $y \in F$ there is some $x \in E$ with $Tx = y$ and $\|x\| \leqslant \left(\frac{R}{1-k}\right)\|y\|$ and (ii) F is complete.

State, *without proof*, the Open Mapping Theorem, for a continuous linear map between two Banach spaces. Deduce the Closed Graph Theorem.

Let X be a compact Hausdorff space and let $C(X)$ be the Banach space of all continuous, real-valued functions on X, equipped with the uniform norm $\|\,.\,\|_\infty$. [You should assume, *without proof*, that $\big(C(X); \|\,.\,\|_\infty\big)$ *is* a Banach space.] For each $x \in X$, let the mapping $\epsilon_x : C(X) \to \mathbb{R}$ be defined by $\epsilon_x(f) = f(x)$ $(f \in C(X))$. Now let $\|\,.\,\|$ be some Banach-space norm on $C(X)$ such that, for every $x \in X$, the mapping ϵ_x is $\|\,.\,\|$-continuous. Prove that $\|\,.\,\|$ is equivalent to $\|\,.\,\|_\infty$.

2 Let E be a real vector space and let K be a convex subset of E. Define what it means to say that a point $x \in K$ is an *extreme point* of K.

Now let E be a real, Hausdorff, locally convex topological vector space and let K be a compact, convex subset of E. Prove that K is the closed convex hull of its set of extreme points. [*You may assume, without proof, any version of the Hahn-Banach theorem, and of the separation theorem for convex sets.*]

Let ℓ^∞ be the space of all bounded real sequences $x = (x_n)_{n \geqslant 1}$ with the norm

$$\|x\|_\infty = \sup\{|x_n| : n \geqslant 1\}\,,$$

and let c_0 be the closed subspace of ℓ^∞ consisting of all sequences $x = (x_n)$ such that $x_n \to 0$ as $n \to \infty$. Identify the extreme points of the closed unit ball of ℓ^∞, and prove that the closed unit ball of c_0 has no extreme points.

3 Let A be a Banach algebra, with identity, over the complex field, and let $x \in A$. Define the *spectrum*, $\operatorname{Sp} x$, of x in A. Prove that $\operatorname{Sp} x$ is a non-empty, compact subset of $\mathbb{C}$. [*You should assume, without proof, that*: (i) *the set $G(A)$ of invertible elements of A is open in A, and* (ii) *the mapping $x \mapsto x^{-1}$ $(x \in G(A))$ is continuous.*]

Let $E = \mathcal{O}(\mathbb{C})$ be the algebra of all complex-valued, holomorphic (i.e. analytic) functions on the complex plane. Show that E can be given an algebra-norm, but not a complete algebra-norm.

Let $D = C(\mathbb{C})$ be the algebra of all complex-valued, continuous functions on the complex plane. Explain, briefly, how to find, in D, a function f and a sequence (g_n) of non-identically-zero functions, such that $fg_n = ng_n$, for all $n \geqslant 1$. Deduce that D can not be given any algebra-norm.

4 Let A be a complex, unital Banach algebra and let $x \in A$. Prove the spectral radius formula,

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geqslant 1} \|x^n\|^{\frac{1}{n}} .$$

[*Elementary properties of the spectrum may be quoted without proof*.]

Prove, from this formula or otherwise, that $r(xy) = r(yx)$ for all x, $y \in A$.

Give an example in which $\mathrm{Sp}(xy) \neq \mathrm{Sp}(yx)$.

Prove also that, for every $x \in A$,

$$r(x) = \inf\{t > 0 : \left(t^{-1}x\right)^n \to 0 \text{ as } n \to \infty\} .$$

5 Let A be a commutative, unital Banach algebra over $\mathbb{C}$ and let $x \in A$. For every open subset U of $\mathbb{C}$, let $\mathcal{O}(U)$ be the algebra of all complex-valued holomorphic functions on U, in its standard topology of 'local uniform convergence'. Let $Z \in \mathcal{O}(U)$ be the function $Z(z) = z$ $(z \in U)$. Prove that there is a continuous, unital homomorphism $\Theta_x : \mathcal{O}(U) \to A$ such that $\Theta_x(Z) = x$, if and only if $\mathrm{Sp}\, x \subset U$. Prove also that Θ_x is unique.

[N.B. *You should assume, without proof, any standard theorems from complex analysis that you need, including any form of the Runge approximation theorem. Any such theorem should be clearly stated.*]

Prove that, for all $f \in \mathcal{O}(U)$ and every character φ on A,

$$\varphi\big(\Theta_x(f)\big) = f\big(\varphi(x)\big) .$$

Deduce that $\mathrm{Sp}\big(\Theta_x(f)\big) = \{f(z) : z \in \mathrm{Sp}\, x\}$.

Let $x \in A$ satisfy $\mathrm{Sp}\, x \subset \Pi_+$, where $\Pi_+ = \{z \in \mathbb{C} : \Re z > 0\}$. Prove that there is a unique element $y \in A$ such that both $y^2 = x$ and $\mathrm{Sp}\, y \subset \Pi_+$.

6 Let A be a complex, unital Banach algebra, with an involution $x \mapsto x^*$ satisfying $\|x^*x\| = \|x\|^2$ $(x \in A)$ (i.e. A is a C^*-*algebra*); we say that $x \in A$ is *normal* if and only if $x^*x = xx^*$. Prove :

(i) if x is a normal element of A, then $r(x) = \|x\|$;

(ii) if A is also commutative, then $\varphi(x^*) = \overline{\varphi(x)}$ for every $x \in A$ and every character φ on A.

Deduce that every commutative C^*-algebra is isometrically $*$-isomorphic to an algebra $C(K)$, of all continuous complex-valued functions on a suitable compact Hausdorff space K.

[*General results on commutative Banach algebras may be quoted without proof*.]

If A is any (not necessarily commutative) C^*-algebra and if x is a normal element of A such that $\operatorname{Sp} x \subset \mathbb{R}$, prove that $x = x^*$. Is the assumption of *normality* necessary for this result ?

END OF PAPER