UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 9 June, 2006 1.30 to 3.30

PAPER 59

QUANTUM INFORMATION, ENTANGLEMENT AND NONLOCALITY

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Alice and Bob possess bit strings $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ respectively; Alice's bit string is known to her but not to Bob, and vice versa. They wish to calculate one bit determined by a Boolean function $f(x_1, \ldots, x_m; y_1, \ldots, y_n)$ of their joint data; the form of f is known to them both. They possess an unlimited supply of "black boxes" which accept one bit inputs x and y from Alice and Bob respectively, and generate one bit outputs a and b for Alice and Bob respectively, with the property that

$$a + b = x \cdot y \text{ (modulo 2)}\,.$$

Each output (a, b) is randomly, independently and equiprobably chosen from the two pairs satisfying this constraint. Show how Alice and Bob can enable Bob to calculate f by combining individual calculations, a series of joint black box invocations, and one bit of classical communication from Alice to Bob.

2 Alice wishes to supply Bob with a copy of a pure state $|\,\psi\,\rangle$, whose classical description she knows. Show that there is a strategy which, with probability 1/2, allows her to succeed in doing this, and for Bob to know this "remote known pure state preparation" succeeded, given one Bell singlet and *one* bit of classical communication.

Show further that there is a strategy which, with probability $1/2^n$, allows Alice to supply Bob with n pure states (known to Alice) $|\,\psi_1\,\rangle, \ldots, |\,\psi_n\,\rangle$, and for Bob to know this has succeeded, using n singlets and one bit of classical communication.

Hence, or otherwise, show that, if Alice and Bob share an unlimited supply of singlets, they can carry out remote known pure state preparation at a classical communication cost that tends asymptotically to one classical bit per remote state.

3 A general quantum measurement is represented by a collection of n measurement operators A_i satisfying the completeness equation

$$\sum_{i=1}^{n} A_i^{\dagger} A_i = I \, .$$

Give the standard expressions for the probability of outcome i and the form of the state after outcome i occurs, if the measurement is carried out on a quantum state described by density matrix ρ.

A projective quantum measurement is a quantum measurement of the above form in which the A_i are all projection operators, which we denote by P_i.

Show that the projection operators defining a projective quantum measurement necessarily commute.

Show also that the form of a general quantum measurement on a system S, together with the expressions for the outcome probabilities and post-measurement states, can be obtained from the corresponding expressions for a projective measurement on a larger system comprising S together with an ancillary system A.

4 Alice and Bob carry out an experiment on a shared physical system in which they each randomly and independently choose one of N possible measurements, drawn from the sets $\{A_1, \ldots, A_N\}$ and $\{B_1, \ldots, B_N\}$ respectively. Their choices are made at spacelike separated points. Each possible measurement has d possible outcomes, taking values in the range $\{0, 1, \ldots, (d-1)\}$. We write $[A_i - B_j]$ to be the difference between the outcomes for the measurement choices A_i and B_j, taken modulo d, so that $[x]$ is always in the range $0, 1, \ldots, (d-1)$. We write $\langle X \rangle$ to be the expected value of X in a given physical theory.

Show that any deterministic local hidden variables theory implies that

$$\sum_{i=1}^{N} \langle [A_i - B_i] \rangle + \sum_{i=1}^{N-1} \langle [B_i - A_{i+1}] \rangle + \langle [B_N - A_1 - 1] \rangle \geqslant d - 1$$

for any set of measurement choices $\{A_i\}$ and $\{B_j\}$.

By explicitly describing sets of quantum measurements and calculating the relevant expectation values in the case $d = 2$, show that quantum theory violates this inequality. Comment briefly on the theoretical implications of this result.

END OF PAPER