UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 13 June, 2006 9 to 11

PAPER 56

SOLITONS AND INSTANTONS

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Find the field equation and the conserved energy for the scalar field theory with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}(\partial_t \phi)^2 - \tfrac{1}{2}(\partial_x \phi)^2 - (1 - \cos\phi).$$

Find the static kink solution interpolating between two vacua

$$\phi(-\infty) = 0, \qquad \phi(\infty) = 2\pi.$$

How many other kink solutions are there?

Let ϕ_0 be a solution to the field equation, and let ϕ_1 satisfy

$$\partial_\rho(\phi_1 - \phi_0) = 2b \sin\left(\tfrac{\phi_1 + \phi_0}{2}\right), \ \partial_\tau(\phi_1 + \phi_0) = 2b^{-1} \sin\left(\tfrac{\phi_1 - \phi_0}{2}\right),$$

where $\tau = (x + t), \rho = (x - t)$ and b is a constant. Show that ϕ_1 is also a solution to the field equation.

2 Let $v = v(x, y) \in \mathbb{R}^n$ be a vector which satisfies a system of equations

$$D_x v := \partial_x v + A_x v = 0, \qquad D_y v := \partial_y v + A_y v = 0, \tag{1}$$

where A_x, A_y are $\mathbf{gl}(n, \mathbb{R})$ valued functions on $\mathbb{R}^2$.

Show that (1) is consistent iff the nonlinear equation

$$\partial_x A_y - \partial_y A_x + [A_x, A_y] = 0 \tag{2}$$

holds. Give the geometric interpretation of (2), and find its most general solution.

Let $(A_i, \Phi) : \mathbb{R}^3 \to \mathbf{gl}(n, \mathbb{R})$ be the Yang–Mills potential and the Higgs field. By considering a symmetry reduction of ASDYM or otherwise, demonstrate that the Bogomolny equations

$$\frac{1}{2}\varepsilon_{ijk} F_{jk} = D_i \Phi$$

admit a Lax representation analogous to (2) but containing a parameter. Here $D_i \Phi = \partial_i \Phi + [A_i, \Phi]$ and $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$.

3 Write an essay on Yang–Mills instantons on $\mathbb{R}^4$.

4 Write down a twistor equation relating points $(\omega^A, \pi_{A'})$ in twistor space $\mathcal{PT}$ to points $x^{AA'}$ in the complexified Minkowski space M_C.

Let $F(\omega^A, \pi_{A'})$ be a patching matrix for a holomorphic vector bundle $\mu : E \to \mathcal{PT}$ with respect to the covering

$$U = \{(\omega^A, \pi_{A'}), \pi_{1'} \neq 0\}, \qquad \tilde{U} = \{(\omega^A, \pi_{A'}), \pi_{0'} \neq 0\}.$$

Assume that E is trivial on twistor lines, and show that on these lines

$$F = \widetilde{H} H^{-1},$$

where H and $\widetilde{H}$ are holomorphic in $\pi_{A'}$ in U and $\tilde{U}$ respectively.

Deduce the existence of a one–form $\Phi_{AA'}$ on M_C such that

$$\pi^{A'} \Phi_{AA'} = H^{-1} \pi^{A'} \frac{\partial}{\partial x^{AA'}} H,$$

and show that this one–form satisfies the ASDYM equations.

END OF PAPER