UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 12 June, 2006 9 to 11

PAPER 55

ADVANCED COSMOLOGY

*Attempt **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In synchronous gauge (with metric perturbations $h^{0\mu}=0$ about a flat FRW universe with $\Omega_{\rm tot}=1$), linear perturbations of a multicomponent fluid obey the following evolution equations

$$\begin{aligned} &\delta_N' + (1+w_N)i\mathbf{k}\cdot\mathbf{v}_N + \tfrac{1}{2}(1+w_N)h' = 0\,,\\ &\mathbf{v}_N' + (1-3w_N)\frac{a'}{a}\mathbf{v}_N + \frac{w_N}{1+w_N}i\mathbf{k}\delta_N = 0\,,\\ &h'' + \frac{a'}{a}h' + 3\left(\frac{a'}{a}\right)^2\sum_N(1+3w_N)\Omega_N\delta_N = 0\,, \end{aligned} \qquad (\dagger)$$

where δ_N is the density perturbation, Ω_N is the fractional density, $\mathbf{v}_N$ is the velocity and $P_N = w_N\rho_N$ is the equation of state of the Nth fluid component, and $\mathbf{k}$ is the comoving wavevector ($k=|\mathbf{k}|$), h is the trace of the metric perturbation and primes denote differentiation with respect to conformal time τ ($d\tau = dt/a$).

(i) Assume that the late universe ($t > t_{\rm eq}$) is filled with two components, (a) comoving non-relativistic matter (cold dark matter) ρ_C with no pressure ($P_C=0$) and (b) a gas of randomly-oriented cosmic strings ρ_S with an average equation of state $P_S = -\rho_S/3$. Show that the cold dark matter–string gas equations arising from ($\dagger$) become

$$\begin{aligned} &\delta_C'' + \frac{a'}{a}\delta_C' - \frac{3}{2}\left(\frac{a'}{a}\right)^2\Omega_C\delta_C = 0\,,\\ &\delta_S'' + 2\frac{a'}{a}\left(\delta_S' - \frac{1}{3}\delta_C'\right) - \frac{1}{3}k^2\delta_S - \left(\frac{a'}{a}\right)^2\Omega_C\delta_C = 0\,. \end{aligned}$$

(ii) By considering a new time variable, $\eta \equiv \rho_S/\rho_C$, show that the dynamical equation for the cold dark matter perturbation δ_C can be re-expressed as

$$\frac{d^2\delta_C}{d\eta^2} + \frac{3+4\eta}{2\eta(1+\eta)}\frac{d\delta_C}{d\eta} - \frac{3}{2\eta^2(1+\eta)}\delta_C = 0\,. \qquad (*)$$

[*Hint:* Recall that $\left(\dfrac{a'}{a}\right)^2 = \dfrac{8\pi G}{3}\rho_{\rm tot}a^2\,, \qquad \dfrac{a''}{a} - \left(\dfrac{a'}{a}\right)^2 = -\dfrac{4\pi G}{3}(\rho_{\rm tot}+P_{\rm tot})a^2\,,$

$\rho_N' + 3\dfrac{a'}{a}(\rho_N + P_N) = 0\,.$]

(iii) Consider early times $\eta \ll 1$ when the cold dark matter dominated over the string component and seek a power series solution of ($*$) of the form $\delta_{\rm C} = a_0\eta^\alpha + a_1\eta^{\alpha+1} + \ldots$. Hence or otherwise show that there is an approximate growing mode solution of the form

$$\delta_C \approx A_{\mathbf{k}}\,\eta\,(1-\tfrac{4}{7}\eta)\,, \qquad (\eta \ll 1)\,.$$

Compare this to the expected growth rate for cold dark matter perturbations in a matter dominated universe.

(iv) Define the Jeans' length $\lambda_{\rm J}$. Now consider solving the cold dark matter–string gas equations in the opposite asymptotic limit $\eta \gg 1$. Show that the cold dark matter perturbation is approximately frozen, $\delta_C \approx$ const. Draw a qualitative diagram of the cold dark matter transfer function $T(k)$ for wavenumbers coming inside the horizon after the time of matter-radiation equality, $t > t_{\rm eq}$. What is the analogue here of the adiabatic initial condition for radiation $\delta_R = \frac{4}{3}\delta_C$ when $k \ll aH$? Briefly discuss the apparent qualitative behaviour of the string perturbations δ_S on both superhorizon and subhorizon scales.

2 (i) Consider a photon with four-momentum p^μ ($p_\mu p^\mu = 0$) propagating in a perturbed FRW universe (flat $\Omega = 1$) with line element

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right]$$

where $\mathbf{k}$ is the comoving wavevector and $\hat{k}^i = k^i/|\mathbf{k}|$. A comoving observer with four-velocity $u^\mu = a^{-1}(1,\,0,\,0,\,0)$ measures the photon energy to be $E = -u_\mu p^\mu = ap^0 \equiv q/a$ where q is the comoving momentum. Use the geodesic equation $\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\nu\sigma}p^\nu p^\sigma = 0$ to show that for a photon trajectory along (unit) direction $\hat{n}^i$ we have to linear order

$$\frac{dq}{d\tau} = -\frac{1}{2}qh'_{ij}\hat{n}^i\hat{n}^j\,, \qquad\qquad \frac{d\hat{n}^i}{d\tau} = \mathcal{O}(h_{ij})\,.$$

[*Hint:* You may assume that $\Gamma^0_{00} = \frac{a'}{a}$, $\Gamma^0_{0i} = 0$, $\Gamma^0_{ij} = \frac{a'}{a}(\delta_{ij} + h_{ij}) + \frac{1}{2}h'_{ij}$, $\Gamma^i_{0j} = \frac{a'}{a}\delta_{ij} + \frac{1}{2}h'_{ij}$ and $\Gamma^i_{jk} = \frac{1}{2}(h_{ij,k} + h_{ik,j} - h_{jk,i})$.]

(ii) The photon distribution function $f(\mathbf{x}, \mathbf{p}, \tau)$ can be expanded about the Planck spectrum $f_0(p,\tau) = f_0(q)$ as

$$f(\mathbf{x}, \mathbf{p}, \tau) = f_0(q) + f_1(\mathbf{x}, q, \hat{\mathbf{n}}, \tau)\,,$$

where the photon momentum $p \equiv q/a$. Show that the collisionless Boltzmann equation

$$\frac{df}{d\lambda} \equiv \frac{dx^\mu}{d\lambda}\frac{\partial f}{\partial x^\mu} + \frac{dp^\mu}{d\lambda}\frac{\partial f}{\partial p^\mu} = 0$$

can be re-expressed in the form

$$\frac{\partial f_1}{\partial \tau} + \hat{n}^i\frac{\partial f_1}{\partial x^i} + \frac{dq}{d\tau}\frac{df_0}{dq} + \frac{dq}{d\tau}\frac{\partial f_1}{\partial q} + \frac{d\hat{n}^i}{d\tau}\frac{\partial f_1}{\partial \hat{n}^i} = 0\,,$$

which, using the results from part (i), at linear order reduces to

$$\frac{\partial f_1}{\partial \tau} + ik\mu f_1 = \frac{1}{2}\frac{df_0}{dq}h'_{ij}\hat{n}^i\hat{n}^j\,,$$

where $\mu = \hat{\mathbf{k}}\cdot\hat{\mathbf{n}}$. Finally, given that ρ_γ is the background photon density, argue that the brightness function

$$\Delta(\mathbf{x}, \hat{\mathbf{n}}, \tau) \equiv 4\frac{\Delta T}{T} \equiv \frac{4\pi}{a^4\rho_\gamma}\int qf_1 q^2 dq$$

must therefore satisfy

$$\Delta' + ik\mu\Delta = -2h'_{ij}\hat{n}^i\hat{n}^j\,. \qquad (\ddagger)$$

(iii) Assume recombination occurs instantaneously at photon decoupling ($\tau = \tau_{\rm dec}$) with the brightness function given in Fourier space by only the two lowest moments,

$$\Delta(\mathbf{k}, \mu, \tau_{\rm dec}) = \delta_\gamma(\mathbf{k}, t_{\rm dec}) + 4\hat{\mathbf{n}}\cdot\mathbf{v}(\mathbf{k}, t_{\rm dec})\,,$$

where δ_γ is the photon density perturbation and $\mathbf{v}$ is the average fluid velocity. Briefly describe the important physical mechanisms which make this a poor approximation on small angular scales (with multipole $\ell < 200$). Use this assumption and eqn ($\ddagger$) to derive the Sachs-Wolfe equation in real space for the temperature fluctuation in a direction $\hat{\mathbf{n}}$:

$$\frac{\Delta T}{T}(\mathbf{x}_0,\,\hat{\mathbf{n}},\,\tau_0) = \frac{1}{4}\delta_\gamma(\mathbf{x}, \tau_{\rm dec}) + \hat{\mathbf{n}}\cdot\mathbf{v}(\mathbf{x},\,\tau_{\rm dec}) - \frac{1}{2}\int_{\tau_{\rm dec}}^{\tau_0} h'_{ij}\hat{n}^i\hat{n}^j d\tau\,.$$

3 In the 3+1 formalism, we split spacetime using the line element

$$ds^2 = -N^2 dt^2 + {}^{(3)}g_{ij}(dx^i - N^i dt)(dx^j - N^j dt)\,,$$

with lapse function $N(t, x^i)$, shift vector $N^i(t, x^i)$ and ${}^{(3)}g_{ij}(x^i)$ the three-metric on constant time spacelike hypersurfaces Σ. The vector $n^\mu = \frac{1}{N}(1,\ N^i)$ is normal to Σ and defines the extrinsic curvature through

$$K_{ij} \equiv -n_{i;j} = -\frac{1}{2N}\left({}^{(3)}g_{ij,0} + N_{i|j} + N_{j|i}\right)\,,$$

where $|$ denotes the covariant derivative in Σ.

(i) Consider the conformal 3-metric

$${}^{(3)}\tilde{g}_{ij} = ({}^{(3)}g)^{-1/3}\ {}^{(3)}g_{ij}$$

where ${}^{(3)}g = \det({}^{(3)}g_{ij})$ and, hence or otherwise, take the trace of the extrinsic curvature expression to find

$$K \equiv {}^{(3)}g^{ij}K_{ij} = -\frac{1}{2N}\left[\frac{{}^{(3)}\dot{g}}{{}^{(3)}g} - 2N^i_{|i}\right]\,.$$

In the context of an expanding universe (setting $N^i = 0$), argue that $-K/3$ can be interpreted as a locally defined Hubble parameter $H(t, x^i)$. [*Hint:* You may assume that $\mathrm{Tr}(A^{-1}dA/dt) = d(\ln(\det A))/dt$ for any matrix A with non-vanishing determinant.]

(ii) When linearising the 3+1 metric about a flat FRW universe, we define the scalar perturbations by

$$N(t, x^i) \equiv \bar{N}(t)(1 + \Phi(t, x^i))\,, \qquad N_i \equiv -a^2 B_{,i}\,, \qquad {}^{(3)}g_{ij} = a^2(t)[(1 - 2\Psi)\delta_{ij} - 2E_{,ij}]\,,$$

and also $\rho = \bar{\rho} + \delta\rho$ and $P = \bar{P} + \delta P$, where bars denote background homogeneous quantities. In synchronous gauge, we take $\Phi = 0$ and $B = 0$. Given that metric perturbations transform as

$$\delta\tilde{g}_{\alpha\beta} = \delta g_{\alpha\beta} - \bar{g}_{\alpha\beta,\gamma}\xi^\gamma - \bar{g}_{\gamma\beta}\xi^\gamma_{,\alpha} - \bar{g}_{\alpha\gamma}\xi^\gamma_{,\beta} \ \text{ under } \ (t,\ x^i) \longrightarrow (\tilde{t},\ \tilde{x}^i) = (t + \xi^0,\ x^i + \xi^i)\,,$$

where $\xi^i \equiv \partial^i\lambda$, show that there is a residual gauge freedom in synchronous gauge given by the coordinate transformation,

$$\xi^0 = \frac{C(x^i)}{\bar{N}}\,, \qquad \lambda = C(x^i)\int\frac{\bar{N}}{a^2}dt + D(x^i)\,,$$

where C and D are arbitrary functions of x^i only. Briefly discuss the significance of this gauge freedom during (a) inflation and (b) the standard hot big bang. In longitudinal Newtonian gauge we take instead $E = B = 0$. Find a transformation law that expresses the density perturbation $\delta\rho/\rho$ in Newtonian gauge in terms of synchronous gauge quantities.

(iii) Show that the quantity

$$\zeta = \Psi - \tfrac{1}{3}\frac{\delta\rho}{\bar{\rho} + \bar{P}}\,,$$

is gauge-invariant and that it is independent of time on superhorizon scales, that is, $\dot{\zeta} = 0$ for $k \ll aH$.

[*Hint:* You may assume a definite equation of state $P = w\rho$, that the perturbed energy density conservation equation is

$$\dot{\delta\rho}/\bar{N} = -3H(\delta\rho + \delta P) + (\bar{\rho} + \bar{P})(\kappa - 3H\Phi) - \triangle u\,,$$

and that the metric perturbation Ψ satisfies $\dot{\Psi}/\bar{N} = -H\Phi + \frac{1}{3}\kappa + \frac{1}{3}\triangle\chi$, where $\triangle \equiv \nabla^2/a^2$, u generates the scalar velocity perturbation, and κ and χ generate the trace and traceless part of K_{ij} respectively.]

END OF PAPER