UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 7 June 2006 1.30 to 4.30

PAPER 53

STRING THEORY

Attempt ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

Minor errors in numerical factors will not be heavily penalized.

The covariant world-sheet action for the bosonic string in flat space-time is

$$I = \frac{-1}{4\pi\alpha'} \int d\sigma d\tau \, \sqrt{-\det\gamma}\, \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \, .$$

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Derive the classical equations of motion and constraints for a closed bosonic string moving in d-dimensional Minkowski space-time.

Define the Virasoro generators and explain how they enter into the definition of physical states.

Show that the massless states of the closed string are given by $\zeta_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0,p\rangle$, where $|0,p\rangle$ is the ground state with momentum p_μ, α^{μ}_{m} and $\tilde{\alpha}^{\mu}_{m}$ are the usual bosonic oscillator modes and the polarization tensor $\zeta_{\mu\nu}$ satisfies conditions that should be derived.

What is the space-time interpretation of the massless closed-string states in critical (26-dimensional) string theory?

How do the space-time fields corresponding to these massless states enter into the functional integral for a closed string moving in a general space-time background?

How does the coupling constant dependence of closed-string perturbation theory arise from the functional integral formulation?

2 How may the constraints of the classical bosonic string be eliminated by use of the light-cone parameterisation?

Deduce the expression for the general excited physical state of a bosonic open string propagating in 26-dimensional Minkowski space-time with Neumann boundary conditions. What is the interpretation of such a string in terms of D-branes?

Suppose there is a circular spatial direction of circumference $2\pi R$. Show that the action of T-duality in this direction changes the boundary conditions from Neumann to Dirichlet.

An open string stretches between two Dp-branes in 26-dimensional Minkowski space separated by a distance d. Determine its ground-state energy and show that its mass becomes tachyonic when $d < d_0$, where d_0 should be determined.

3 Explain how the local symmetries of the bosonic string can be used to choose a parameterisation of the world-sheet so that the action reduces to that of d free bosons, at least on a world-sheet of spherical topology.

Show how this leads to the ghost action

$$I^{(b,c)} = -\frac{i}{2\pi}\int d\sigma d\tau\, c^{\alpha}\,\partial^{\beta}\,b_{(\alpha\beta)}\,,$$

where the components of the vector $c^{\alpha}(\sigma,\tau)$ are the anticommuting Faddeev–Popov ghosts and the components of the symmetric, traceless second-rank tensor, $b_{(\alpha\beta)}(\sigma,\tau)$, are the antighosts. [*You may use the fact that*

$$\int Db\,Dc\,\exp\Big(-\frac{1}{2\pi}\int d\sigma d\tau\, b_{\alpha\beta}\,\mathcal{P}^{(\alpha\beta)}_{\delta}\,c^{\delta}\Big) = \det\mathcal{P}\,,$$

where $\mathcal{P}^{(\alpha\beta)}_{\delta}$ *is an operator that maps two-vectors into symmetric, traceless second-rank tensors.*]

Show further that this action can be written in world-sheet light-cone components as

$$I^{(b,c)} = \frac{i}{\pi}\int d\sigma d\tau\,(c^{+}\,\partial_{-}b_{++} + c^{-}\,\partial_{+}b_{--})\,,$$

where $\pm$ denote light-cone components (defined so that $v^{\pm} = v^{0} \pm v^{1}$, for any vector v^{α}).

Show that the total action, $I + I^{(b,c)}$ is invariant under the fermionic BRST transformations,

$$\delta_{\eta}X^{\mu} = \eta\,(c^{+}\,\partial_{+}X^{\mu} + c^{-}\,\partial_{-}X^{\mu})\,, \qquad \delta_{\eta}c^{+} = \eta\,c^{+}\,\partial_{+}c^{+}\,, \qquad \delta_{\eta}c^{+} = \eta\,c^{-}\,\partial_{-}c^{-}\,,$$

$$\delta_{\eta}b_{++} = \frac{i}{\alpha'}\eta\,\theta_{++}\,, \qquad \delta_{\eta}b_{--} = \frac{i}{\alpha'}\eta\,\theta_{--}\,,$$

where η is a constant Grassmann parameter and $\theta_{\alpha\beta}$ is the energy-momentum tensor, which has components $\theta_{+-} = 0$ and

$$\theta_{++} = \partial_{+}X^{\mu}\partial_{+}X_{\mu} - i\alpha'\left[c^{+}\partial_{+}\,b_{++} + 2\partial_{+}c^{+}\,b_{++}\right]\,,$$

with a similar expression for θ_{--}.

Deduce that the BRST transformation is nilpotent, i.e., two successive transformations with parameters η_1 and η_2 annihilate any field so that

$$\delta_{\eta_1}\delta_{\eta_2}\,\Phi = 0\,,$$

where Φ is any of the world-sheet fields, X^{μ}, b or c. [*You may use, without proof, the fact that* $\delta_{\eta}\,\theta_{\alpha\beta} = 0$.]

4 The action for the open fermionic string propagating in 10-dimensional Minkowski space-time can be written in the form (in conformal gauge on a lorentzian world-sheet and with $\alpha' = 1/2$)

$$I = \frac{1}{\pi}\int d\sigma d\tau \; (2\partial_+ X^\mu \partial_- X_\mu + i\psi_2^\mu \partial_+ \psi_{2\mu} + i\psi_1^\mu \partial_- \psi_{1\mu}) \; ,$$

where $\psi_a^\mu(\sigma,\tau)$ ($\mu = 0, 1, \ldots, 9$) are 10 Majorana world-sheet spinors with spinor index $a = 1, 2$, and $\partial_\pm = (\partial_\tau \pm \partial_\sigma)/2$. Show that this action is invariant under the two-dimensional supersymmetry transformations

$$\delta_\epsilon X^\mu = -\frac{i}{2}\epsilon_1 \psi_1^\mu - \frac{i}{2}\epsilon_2 \psi_2^\mu \, , \qquad \delta_\epsilon \psi_1^\mu = \partial_+ X^\mu \epsilon_1 \, , \qquad \delta_\epsilon \psi_2^\mu = \partial_- X^\mu \epsilon_2 \, , \qquad (*)$$

where ϵ_1 and ϵ_2 are the Grassmann-valued components of a constant world-sheet Majorana spinor.

The supercurrent, J_α^a, has components

$$J_+^1 = \psi_1^\mu \partial_+ X_\mu \, , \qquad J_-^2 = \psi_2^\mu \partial_- X_\mu \, , \qquad J_+^2 = J_-^1 = 0 \, .$$

Show that the supercharge, $q_\epsilon = \frac{1}{\pi}\int d\sigma(\epsilon_1 J_+^1 + \epsilon_2 J_-^2)$, generates the supersymmetry transformations $(*)$ i.e., for any world-sheet field $\Phi(\sigma,\tau)$,

$$\delta_\epsilon \Phi = [q_\epsilon, \Phi] \, .$$

[*You may assume the equal-τ (anti)commutation relations,*

$$[\partial_\tau X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = -i\pi\delta(\sigma - \sigma')\eta^{\mu\nu} \, , \qquad \{\psi_i^\mu(\sigma,\tau), \psi_j^\nu(\sigma',\tau)\} = \pi\eta^{\mu\nu}\delta_{ij}\delta(\sigma - \sigma')$$

where $\eta^{\mu\nu}$ is the d-dimensional Minkowski metric.]

Using the (anti)commutation relations show that the anticommutator of two supercurrents can be written in the form

$$\{J_+^1(\sigma,\tau), J_+^1(\sigma',\tau)\} = \pi\delta(\sigma - \sigma')\,\theta_{++}(\sigma,\tau) \, , \qquad (*)$$

and hence determine the form of the energy-momentum tensor, θ_{++}.

Indicate, *without proof and without detailed coefficients*, the structure of the other (anti)commutation relations involving J_+ and θ_{++} that, together with $(*)$, form a closed algebra.

How are physical states of the closed fermionic string defined in terms of the generators of this algebra?

END OF PAPER