UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 7 June, 2006 9 to 12

PAPER 52

THE STANDARD MODEL

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write an essay on the problem of introducing massive vector bosons into quantum field theory.

2 Explain, with the aid of a Z_2-symmetric theory, how spontaneous symmetry breaking occurs in a quantum field theory.

Write down the formula for the partial width Γ of an on-shell particle of mass m that decays to two on-shell particles, in terms of the matrix element $\mathcal{M}$.

The Feynman rule for a vertex coupling a Higgs to W^+W^- is $igM_W g_{\mu\nu}$. Assuming $M_H > 2M_Z > 2M_W$, calculate the partial width for the decay $H \to W^+W^-$ in terms of the Higgs mass M_H, $x = M_W/M_H$ and $G_F = \sqrt{2}g^2/8M_W^2$.

From this answer, write down the analogous partial width of the decay $H \to ZZ$ in terms of $y = M_Z/M_H$.

[*You may assume, for summations over the W polarisation states λ, that*

$$\sum_\lambda \epsilon_\mu^*(p,\lambda)\epsilon_\nu(p,\lambda) = -g_{\mu\nu} + \frac{p^\mu p^\nu}{M_W^2}. \qquad]$$

3 Let C be an anti-symmetric matrix such that $C^{-1}\gamma^\mu C = -(\gamma^\mu)^t$, where $CC^\dagger = 1$ and t denotes the transpose. Show that if $\psi(x)$ satisfies the free Dirac equation then so does $C(\bar{\psi}(x))^t$. Given that $\hat{C}\psi(x)\hat{C}^{-1} = \eta_C C\bar{\psi}(x)^t$, find how $\bar{\psi}(x)$ transforms under charge conjugation $\hat{C}$.

How does one expect creation and annihilation operators to behave under $\hat{C}$? By considering the expansion of Dirac fields in terms of creation and annihilation operators, show that

$$v(p,s) = C\bar{u}(p,s)^t,$$

where $v(p,s)$ and $u(p,s)$ are positron and electron spinors respectively.

How does the vector current $j_\mu = \bar{\psi}\gamma_\mu\psi$ transform under $\hat{C}$? What implications does this have for the photon field A_μ? Show that $C^{-1}\gamma_5 C = \gamma_5^t$ and hence determine the behaviour of the axial current $j_{5\mu} = \bar{\psi}\gamma_\mu\gamma_5\psi$ under $\hat{C}$. What implication do the above results have for the $\hat{C}$ invariance or non-invariance of a $(j_\mu - j_{5\mu})A^\mu$ type interaction?

Under a parity transformation $\hat{P}$, $\hat{P}\psi(x)\hat{P}^{-1} = \eta_P\gamma^0\psi(x_P)$, where $x_P^\mu = (t, -\mathbf{x})$. How do the vector and axial currents transform under $\hat{P}$?

How do $(j_\mu - j_{5\mu})V^\mu$ interactions transform under the combined action of $\hat{C}\hat{P}$ if V^μ is a real vector field? What does this imply about the interactions of the Z boson in the Standard Model?

[*You may find the fact that* $\gamma_0\gamma_0^\dagger = 1$ *useful.*]

4 Justify the fact that the renormalised coupling in a quantum field theory $g(\mu^2)$ is a function of the renormalisation scale μ and show that $\alpha = g^2/4\pi$ satisfies a renormalisation group equation of the form

$$\frac{d\alpha}{d(\ln \mu^2)} = b_0\alpha^2 + O(\alpha^3)$$

where b_0 is a real constant.

Dropping the $O(\alpha^3)$ terms, solve the equation above subject to the boundary condition $\alpha(\mu_0^2) = \alpha_0$. If $b_0 = -\beta_0$, where β_0 is a positive constant, show that the solution may be rewritten as

$$\alpha(\mu^2) = \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2)} \ , \qquad (*)$$

and determine $\ln \Lambda^2$ in terms of μ_0, α_0^{-1} and β_0. Explain what happens as $\mu^2 \to \infty$, and also what happens as μ^2 becomes small, and briefly discuss the implications for strong interaction physics. Roughly what size should Λ be, assuming β_0 is of order unity?

The explicit expression for β_0 in QCD is

$$\beta_0 = \frac{11N - 2n_f}{12\pi}$$

where N is the number of colours and n_f is the number of flavours of quark Dirac spinors. Calculate β_0 for the QCD coupling in the Standard Model.

Quarks that have masses larger than μ are not included in the effective field theory. As μ increases across a quark mass threshold m_q, $n_f \to n_f + 1$. In the solution $(*)$, Λ_{n_f} depends on n_f in such a way that $\alpha(\mu^2)$ is continuous at a quark mass threshold. Show that if Λ_5 appears in the formula $(*)$ for the coupling for $\mu^2 > m_b^2$ and Λ_4 for $\mu^2 < m_b^2$, where m_b is the bottom quark mass, then

$$\Lambda_5 = \Lambda_4 \left(\frac{m_b}{\Lambda_4}\right)^{-2/23} .$$

END OF PAPER