UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 8 June, 2006 9 to 11

PAPER 51

STATISTICAL FIELD THEORY

Attempt ***TWO*** *questions.*

There are ***THREE*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Give an account of the Landau-Ginsberg (LG) theory of phase transitions which should include a discussion of the following topics:

(a) The idea of an order parameter;

(b) The correlation length;

(c) The distinction between first-order and continuous phase transitions and how their occurrence is predicted in LG theory;

(d) The concept of universality and which properties are, and are not, universal at a critical point;

(e) The idea of *critical exponents* and how they may be derived;

(f) The features of a tricritical point, how it occurs in LG theory and a brief description of a phase diagram containing a tricritical point.

You should clarify your account with diagrams which should be appropriately labelled.

Explain briefly the idea of the mean field approach and why it is only likely to be valid in limit of large dimension, D.

The Ising model in D dimensions is defined on a cubic lattice Λ with N sites and with spin $\sigma_{\mathbf{n}} \in \{1, -1\}$ on the site at position $\mathbf{n}$. The Hamiltonian with zero external magnetic field is

$$\mathcal{H}(\{\sigma\}) \;=\; -J \sum_{\mathbf{n}\in\Lambda,\mu} \sigma_{\mathbf{n}}\sigma_{\mathbf{n}+\mu},$$

where the sum on μ is over the basis vectors of the lattice, Λ. Using the mean-field approximation show that the magnetisation M satisfies the equation

$$M \;=\; \tanh(2DJM/k_BT)\;,$$

where k_B is Boltzmann's constant and T is the temperature. Hence, or otherwise, show that in this case mean-field theory predicts a second order phase transition at $T = 2DJ/k_B$.

2 The Ising model in D dimensions is defined on a cubic lattice of spacing a with N sites and with spin σ_r on the r-th site. The Hamiltonian is defined in terms of a set of operators $O_i(\{\sigma\})$ by

$$\mathcal{H}(\mathbf{u},\sigma) \;=\; \sum_i u_i O_i(\{\sigma\}) \;,$$

where the u_i are coupling constants with $\mathbf{u} = (u_1, u_2, \ldots)$ and $\sigma_r \in \{1, -1\}$. In particular, $\mathcal{H}$ contains the term $-h\sum_r \sigma_r$ where h is the magnetic field. The partition function is given by

$$\mathcal{Z}(\mathbf{u}, C, N) \;=\; \sum_\sigma \exp(-\beta\mathcal{H}(\mathbf{u},\sigma) - \beta N C) \;.$$

Define the two-point correlation function $G(\mathbf{r})$ for the theory and state how the correlation length ξ parametrizes its behaviour for $r \gg \xi$.

State how the magnetization M and the magnetic susceptibility χ can be determined from the free energy F, and derive the relation which expresses χ in terms of $G(\mathbf{r})$.

Explain how a renormalization group (RG) transformation may be defined in terms of a blocking kernel which, after p iterations, yields a blocked partition function $\mathcal{Z}(\mathbf{u}_p, C_p, N_p)$. Why does $\mathcal{Z}(\mathbf{u}_p, C_p, N_p)$ predict the same large-scale properties of the system as does $\mathcal{Z}(\mathbf{u}, C, N)$?

Derive the RG equation for the free energy $F(\mathbf{u}_p, C_p)$ and explain how it may be expressed in terms of a singular part $f(\mathbf{u})$ which obeys the RG equation

$$f(\mathbf{u}_0) \;=\; b^{-pD} f(\mathbf{u}_p) + \sum_{j=0}^{p-1} b^{-jD} g(\mathbf{u}_j) \;, \qquad (*)$$

where the rôle of the function $g(\mathbf{u})$ should be explained.

Explain the ideas of a fixed point, a critical surface, relevant and irrelevant operators, and a repulsive trajectory in the context of the RG equations. Sketch some typical RG flows near to a critical surface.

Show how the critical exponents characterizing a continuous phase transition may be derived, and explain under what conditions the second term (the inhomogeneous term) on the right-hand-side of $(*)$ may be safely neglected.

In the case that there are two relevant couplings $t = (T - T_C)/T_C$ and h, derive the scaling hypothesis for the singular part F_s of the free energy

$$F_s \;=\; |t|^{D/\lambda_t} f_\pm\left(\frac{h}{|t|^{\lambda_h/\lambda_t}}\right) \;,$$

What is the significance of the subscript label $\pm$ on these functions?

Question 2 continues on the next page

2 (continued) The following critical exponents are defined:

$$\begin{array}{llll} \xi & \sim & |t|^{-\nu} & h=0 \\ M & \sim & |t|^{\beta} & h=0,\ T<T_C \\ \chi & \sim & |t|^{-\gamma} & h=0 \\ C_V & \sim & |t|^{-\alpha} & h=0 \text{ (the specific heat)} \\ M & \sim & |h|^{1/\delta} & T=T_C \end{array}$$

Under suitable assumptions derive the relations

$$\alpha + 2\beta + \gamma \;=\; 2\,, \quad \alpha \;=\; 2 - D\nu\,, \quad \beta\delta \;=\; \beta + \gamma\,,$$

where D is the dimension of space.

According to the scaling hypothesis the correlation function takes the form

$$G(\mathbf{r}) \;=\; \frac{1}{|\mathbf{r}|^{D-2+\eta}}\, f_G\left(\frac{|\mathbf{r}|}{\xi(t,h)}\right)\,.$$

What behaviour is expected for the function $f_G(x)$ in the limit $x \to \infty$?

From this formula obtain an expression for the susceptibility and derive the identity

$$\gamma \;=\; (2-\eta)\nu\,.$$

3 Explain what is meant by the *partition function*, Z, of a statistical system and state how Z is related to the *free energy*, F.

A system in D dimensions is described by a Hamiltonian density $\mathcal{H}(\Lambda, \phi)$, where ϕ is a scalar field and Λ is the Ultra-Violet cut-off in units of inverse length. Explain how the dependence of $\mathcal{H}$ on Λ can be such that the large-scale properties of the system are independent of Λ. Why is it advantageous to do this? Hence show that, under certain assumptions that should be stated, Landau's approach corresponds to the identification

$$F(M) \;=\; \lim_{\Lambda \to 0} \mathcal{H}(\Lambda, M)\,,$$

where the meaning of the field M is to be defined.

Give a reason why it is expected that the predictions from Landau theory for critical exponents break down at and below a critical dimension, D_c. Calculate D_c for an ordinary critical point in a system described by a scalar field. You may assume the details of the loop expansion.

END OF PAPER