UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 5 June, 2006 9 to 12

PAPER 50

SYMMETRY AND PARTICLE PHYSICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
10 sheets of triangular graph paper

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\mathcal{L}$ be a Lie algebra. Define the adjoint representation ad of $\mathcal{L}$ acting on $\mathcal{L}$ and prove that ad is a representation. Define the Killing form κ of $\mathcal{L}$ and prove that

$$\kappa([X,Y],Z) = \kappa(X,[Y,Z])$$

for $X, Y, Z \in \mathcal{L}$. Let $c_{ab}{}^{d}$ be the structure constants of $\mathcal{L}$, and let $c_{abc} \equiv c_{ab}{}^{d}\kappa_{cd}$. Show that c_{abc} is totally antisymmetric in a, b, c.

Suppose that $\mathcal{L}$ is simple and of compact type. Prove that the adjoint representation ad is irreducible. Let d be an anti-hermitian irreducible representation of $\mathcal{L}$ acting on a vector space V ($\dim V > 1$). Define

$$H(X,Y) = \mathrm{Tr}\,(d(X)d(Y))$$
$$B(X,Y,Z) = \mathrm{Tr}\,(d(X)d(Y)d(Z))$$

for $X, Y, Z \in \mathcal{L}$. Show that

$$H([X,Y],Z) = H(X,[Y,Z])$$

and prove that in an adapted basis for $\mathcal{L}$ in which $\kappa_{ab} = -\delta_{ab}$

$$H_{ab} = -\mu\delta_{ab}$$

for constant μ. Show that $\mu > 0$.

By considering $B([X,Y],Z,W)$, prove the identity

$$c_{da}{}^{\ell}B_{bc\ell} + c_{db}{}^{\ell}B_{ca\ell} + c_{dc}{}^{\ell}B_{ab\ell} = 0\ .$$

Also show that B satisfies the identity

$$c_a{}^{mn}B_{mnb} = -\frac{1}{2}\mu\delta_{ab}$$

where indices are raised with the inverse of the Killing form.

[*You may use without proof Schur's Lemmas.*]

2 The non-vanishing commutation relations for a finite dimensional antihermitian representation d of the complexified $\mathcal{L}(SU(3))$ algebra are

$$[H_1, E^1_\pm] = \pm E^1_\pm \ , \qquad [H_1, E^2_\pm] = \mp\frac{1}{2}E^2_\pm \ , \qquad [H_1, E^3_\pm] = \pm\frac{1}{2}E^3_\pm \ ,$$

$$[H_2, E^1_\pm] = 0 \ , \qquad [H_2, E^2_\pm] = \pm\frac{\sqrt{3}}{2}E^2_\pm \ , \qquad [H_2, E^3_\pm] = \pm\frac{\sqrt{3}}{2}E^3_\pm \ ,$$

and

$$[E^1_+, E^1_-] = H_1 \ , \quad [E^2_+, E^2_-] = \frac{\sqrt{3}}{2}H_2 - \frac{1}{2}H_1 \ , \quad [E^3_+, E^3_-] = \frac{\sqrt{3}}{2}H_2 + \frac{1}{2}H_1 \ ,$$

and

$$[E^1_+, E^2_+] = \frac{1}{\sqrt{2}}E^3_+ \ , \qquad [E^1_-, E^2_-] = -\frac{1}{\sqrt{2}}E^3_- \ ,$$
$$[E^1_+, E^3_-] = -\frac{1}{\sqrt{2}}E^2_- \ , \quad [E^1_-, E^3_+] = \frac{1}{\sqrt{2}}E^2_+ \ ,$$
$$[E^2_+, E^3_-] = \frac{1}{\sqrt{2}}E^1_- \ , \qquad [E^2_-, E^3_+] = -\frac{1}{\sqrt{2}}E^1_+ \ ,$$

where $iH_1, iH_2, i(E^m_+ + E^m_-), E^m_+ - E^m_-$ are the antihermitian $\mathcal{L}(SU(3))$ generators in the representation d. Show how this algebra contains three complexified $\mathcal{L}(SU(2))$ algebras. Define the weights associated with this representation, and determine the effect of the raising and lowering operators on the weights.

Suppose that d is irreducible. Define the highest weight state of the representation d. Draw examples of the different types of weight diagrams which can arise. Describe the symmetries of these diagrams, and how the form of the diagrams depends on the location of the highest weight. State (without proof) the rules governing the multiplicities of states in the diagrams.

(i) State the weights associated with the u, d, s quark states in the fundamental representation **3**, and also compute the weights associated with the adjoint representation. Draw the weight diagrams.

(ii) Using $\mathcal{L}(SU(3))$ weight diagrams, derive the decomposition of the tensor product $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$ into $\mathcal{L}(SU(3))$ irreducible representations. [It is not necessary to find explicit expressions for the wave functions in terms of u, d, s.]

(iii) Using $\mathcal{L}(SU(3))$ weight diagrams, decompose $\mathbf{3} \otimes \mathbf{8}$ into $\mathcal{L}(SU(3))$ irreducible representations.

[*You may assume without proof any properties of representations of* $\mathcal{L}(SU(2))$.]

3 Let G be a Lie group with Lie algebra $\mathcal{L}(G)$. Let D_μ be the adjoint covariant derivative acting on $\Phi \in \mathcal{L}(G)$ via

$$D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi]$$

where A_μ is the Yang-Mills potential. How does A_μ transform under the action of G?

The Yang-Mills field strength F is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \ .$$

Prove that F transforms as $F \to F' = gFg^{-1}$ under the action of G. Show that F satisfies the Bianchi identity

$$D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0 \ .$$

Suppose that $\mathcal{L}(G)$ is semi-simple with Killing form κ. Define the Yang-Mills action

$$S_{YM} = \frac{1}{4e^2} \int d^4x \ \kappa(F_{\mu\nu}, F^{\mu\nu}) \ .$$

Derive the Yang-Mills field equations. Consider the θ-term action given by

$$S_\theta = \theta \int d^4x \ \epsilon^{\mu\nu\rho\sigma} \kappa(F_{\mu\nu}, F_{\rho\sigma})$$

for constant θ. Determine the field equations of the action $S = S_{YM} + S_\theta$.

Suppose that F satisfies the Yang-Mills equations. Show that F satisfies

$$D^\rho D_\rho F_{\mu\nu} = 2[F_\mu{}^\rho, F_{\nu\rho}] \ .$$

[*For the purposes of this question, you may assume the identity*

$$\kappa([X,Y],Z) = \kappa(X,[Y,Z]) \ .]$$

4 Give an account of the theory of unitary representations of the Poincaré group.

You should prove how the Pauli-Lubanski vector W_μ of a representation d of the Poincaré algebra, and the "little group", can be used to classify the representation in terms of timelike or null 4-momenta of physical interest, giving a detailed description of both cases.

[*For the purposes of this question, you may assume without proof the Poincaré algebra:*

$$\begin{aligned}
[M^{\mu\nu}, M^{\rho\sigma}] &= i\big(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}\big)\\
[P^\mu, M^{\rho\sigma}] &= i\eta^{\rho\mu}P^\sigma - i\eta^{\sigma\mu}P^\rho\\
[P^\mu, P^\nu] &= 0
\end{aligned}$$

and the following commutation relations involving W

$$[W_\mu, d(P_\nu)] = 0, \qquad [W_\mu, d(M_{\rho\sigma})] = i\eta_{\mu\rho}W_\sigma - i\eta_{\mu\sigma}W_\rho \ .$$

You may also assume that a unitary representation $\mathcal{D}$ *of the Poincaré group acting on a vector space* V *gives rise to a representation* d *of the Poincaré algebra related via*

$$\mathcal{D}\big(e^{-\frac{i}{2}(b_\mu P^\mu + \omega_{\mu\nu}M^{\mu\nu})}\big) = e^{-\frac{i}{2}(b_\mu d(P^\mu) + \omega_{\mu\nu}d(M^{\mu\nu}))}$$

for real b_μ *and skew-symmetric real* $\omega_{\mu\nu}$*, and* $d(M_{\rho\sigma})$ *and* $d(P_\mu)$ *are hermitian.*]

END OF PAPER