UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 6 June, 2006 9 to 12

PAPER 49

ADVANCED QUANTUM FIELD THEORY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Assuming

$$\langle q'|e^{-i\frac{1}{2}\hat{p}^2 T}|q\rangle = \left(\frac{1}{2\pi i T}\right)^{\frac{1}{2}} e^{i\frac{(q'-q)^2}{2T}} \,,$$

where $\hat{q}, \hat{p}$ are the usual quantum mechanical position, momentum operators with $\hat{q}|q\rangle = q|q\rangle$, obtain the path integral representation involving a functional integral over paths $q(t)$ with $q(0) = q\,,\ q(T) = q'$,

$$\langle q'|e^{-i\hat{H}T}|q\rangle = \int \mathrm{d}[q]\, e^{iS[q]} \,,$$

where the Hamiltonian $\hat{H}$ and the action $S[q]$ are given by

$$\hat{H} = \tfrac{1}{2}\,\hat{p}^2 + V(\hat{q})\,, \qquad S[q] = \int_0^T \mathrm{d}t \left(\tfrac{1}{2}\,\dot{q}^2 - V(q)\right) .$$

For $V(q)$ quadratic in q, show that

$$\langle q'|e^{-i\hat{H}T}|q\rangle = N\, e^{iS[q_c]} \,,$$

where q_c is the classical path linking q and q' and N is independent of q, q'. Show that N can be expressed in terms of the determinant of the operator $\frac{\mathrm{d}^2}{\mathrm{d}t^2} + V''(q_c)$ acting on functions $f(t)$ with $f(0) = f(T) = 0$.

Consider the action

$$S[q] = \int_{-\infty}^{\infty} \mathrm{d}t \left(\tfrac{1}{2}\,\dot{q}^2 - \tfrac{1}{2}m^2 q^2 + Jq\right) ,$$

for arbitrary $J(t)$. Show that the functional integral over appropriate paths $q(t)$ with $-\infty < t < \infty$ can in this case be expressed the form

$$\int \mathrm{d}[q]\, e^{iS[q]} = N\, e^{-\frac{1}{2}i\int \mathrm{d}t\mathrm{d}t'\, J(t)G(t-t')J(t')} \,, \quad G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, e^{i\omega t}\, \frac{1}{\omega^2 - m^2 + i\epsilon} \,.$$

2 For a scalar field theory in d-dimensions with Lagrangian density

$$\mathcal{L}(\phi) = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{24}\,\lambda\phi^4\,,$$

define the generating functional $Z[J]$ and also the corresponding functionals $W[J]$ and $\Gamma[\varphi]$ where $\delta W/\delta J = \varphi$. Assume that for $J=0$ then $\varphi = 0$.

Let

$$\int \mathrm{d}^d x_1 \ldots \mathrm{d}^d x_n \, e^{ip_1\cdot x_1 + \ldots + p_n \cdot x_n} \, \frac{\delta^n \Gamma[\varphi]}{\delta\varphi(x_1)\ldots\delta\varphi(x_n)}\bigg|_{\varphi=0}$$
$$= (2\pi)^d \delta^d(p_1 + \ldots + p_n)\, \tau_n(p_1, \ldots, p_n)\,.$$

Show that, for $\lambda = 0$, $\tau_2(p,-p) = p^2 + m^2$ and $\tau_n = 0$ for $n \geq 3$. If $\lambda > 0$, show that for contributions corresponding to Feynman diagrams with zero loops τ_2 is unchanged but $\tau_4(p_1,p_2,p_3,p_4) = \lambda$ and we may take $\Gamma[\varphi] = -\int \mathrm{d}^d x\, \mathcal{L}(\varphi)$.

Draw the Feynman diagrams representing the one loop contributions to τ_2 and τ_4. Evaluate the divergent part of the one loop contribution to τ_2 for $d \approx 4$ and show that it can be cancelled by an additional counterterm $\mathcal{L}_{\text{c.t.}} = -\frac{1}{2}B\phi^2$ for suitable B.

[You may assume $\int \mathrm{d}^d k\, e^{-\alpha k^2} = (\pi/\alpha)^{\frac{1}{2}d}$ for $\alpha > 0$ and $k^2 = \sum_{i=1}^d k_i^2$, and also $\int_0^\infty \mathrm{d}s\, s^{a-1}e^{-s} = \Gamma(a)$ defines the standard Gamma function.]

3 Consider a quantum field theory with a single scalar field and a Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2 - \sum_{n\geq 3}\frac{1}{n!}\,g_n\phi^n\,.$$

In d spacetime dimensions what is the dimension of ϕ? Show that the coupling g_n has dimension $d_n = n - d(\tfrac{1}{2}n - 1)$. For a Feynman diagram with L loops, I internal lines, E external lines and V_n vertices with n lines incident show that the overall degree of divergence of the Feynman integral can be expressed as

$$D = d - (\tfrac{1}{2}d - 1)E - \sum_n d_n V_n\,.$$

Comment on the distinction between renormalisable and nonrenormalisable theories for $d = 3, 4$. Why does considering d to be an arbitrary complex parameter provide a prescription for regularising the theory?

Suppose $g_n = 0$ for $n \neq 4$ and $g_4 = \mu^\varepsilon\lambda$ with λ a dimensionless coupling, $\varepsilon = 4 - d$, μ an arbitrary mass scale. Describe in outline how the functional integral with dimensional regularisation leads to a finite perturbation expansion for the four dimensional quantum field theory in terms of the action

$$S_0 = \int \mathrm{d}^d x\,\mathcal{L}_0\,, \qquad \mathcal{L}_0 = -\tfrac{1}{2}(\partial\phi_0)^2 - \tfrac{1}{2}m_0^2\phi_0^2 - \tfrac{1}{24}\lambda_0\phi_0^4\,,$$

where $\phi_0 = Z^{\frac{1}{2}}\phi$, $m_0^2 = Z_{m^2}m^2$, $\lambda_0 = \mu^\varepsilon Z_\lambda\lambda$. Here Z, Z_{m^2}, Z_λ depend on λ, ε with $Z, Z_{m^2}, Z_\lambda = 1 + \mathrm{O}(\lambda)$. It is sufficient to indicate why $F(\lambda, m^2, h; \mu)$ defined by

$$e^{iF(\lambda,m^2,h;\mu)} = \frac{1}{N_0}\int \mathrm{d}[\phi]\,e^{iS_0[\phi] + \mu^{-\frac{1}{2}\varepsilon}h\int \mathrm{d}^d x\,\phi}\,,$$

where N_0 is chosen so that $F = 0$ for $h = 0$, is a finite function of λ, m^2, h. Explain why

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}F = 0 \quad \text{for constant} \quad \lambda_0, m_0^2, h_0 \quad \text{where} \quad h_0 = \mu^{-\frac{1}{2}\varepsilon}Z^{-\frac{1}{2}}h\,.$$

Hence obtain the RG equation

$$\left(\mu\frac{\partial}{\partial\mu} + \hat{\beta}_\lambda\frac{\partial}{\partial\lambda} + \gamma_{m^2}m^2\frac{\partial}{\partial m^2} + \hat{\gamma}_h h\frac{\partial}{\partial h}\right)F(\lambda, m^2, h; \mu) = 0\,,$$

for suitable $\hat{\beta}_\lambda, \gamma_{m^2}, \hat{\gamma}_h$ depending on λ.

Why may we write $F(\lambda, m^2, h; \mu) = \hat{F}(\lambda, m^2/\mu^2, h/\mu^3)$? Show how the RG equation may be solved for $m^2 = 0$ in terms of a suitable $\lambda(\mu), h(\mu)$. Explain what is meant by a UV fixed point.

4 Explain why gauge fixing is necessary in order to obtain a perturbative expansion for the quantum field theory for a non abelian gauge theory. Suppose the Lagrangian density after gauge fixing has the form

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}{}_a F_{\mu\nu a} + b_a \partial^\mu A_{\mu a} + \tfrac{1}{2}\xi\, b_a b_a + \bar{c}_a \partial^\mu (D_\mu c)_a\,,$$
$$F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + f_{abc} A_{\mu b} A_{\nu c}\,, \quad (D_\mu c)_a = \partial_\mu c_a + f_{abc} A_{\mu b} c_c\,,$$

where $A_{\mu a}$ is the gauge field with group index a, $\bar{c}_a, c_a$ are ghost fields, b_a is an auxiliary scalar field and f_{abc} are the antisymmetric structure constants for the associated Lie algebra. Show that $s\mathcal{L} = 0$ and $s^2 = 0$ where the action of s is defined by

$$sA_{\mu a} = (D_\mu c)_a\,, \qquad sc_a = -\tfrac{1}{2} f_{abc} c_b c_c\,, \qquad s\bar{c}_a = -b_a\,, \qquad sb_a = 0\,,$$

and also $s(c_a X) = (sc_a)X - c_a\, sX$ and similarly for $c \to \bar{c}$.

Assume Q_{BRS}, satisfying ${Q_{BRS}}^2 = 0$, ${Q_{BRS}}^\dagger = Q_{BRS}$, is the corresponding operator acting on states in the quantum field theory. Explain how a space of physical states is constructed. Suppose $|A_{\mu a}(k)\rangle$, $|c_a(k)\rangle$, $|\bar{c}_a(k)\rangle$ are single particle states with 4-momentum k and with non zero scalar products

$$\langle A_{\mu a}(k)|A_{\nu b}(k')\rangle = \eta_{\mu\nu}\delta_{ab}\,\delta_{kk'}\,, \qquad \langle \bar{c}_a(k)|c_b(k')\rangle = \delta_{ab}\,\delta_{kk'}\,.$$

If

$$Q_{BRS}|A_{\nu a}(k)\rangle = \alpha k_\nu |c_a(k)\rangle\,, \qquad Q_{BRS}|\bar{c}_a(k)\rangle = \bar{\alpha} k^\mu |A_{\mu a}(k)\rangle\,,$$

what are the conditions following from BRS symmetry. Determine the physical single particle states and show that they have positive norm.

END OF PAPER