UNIVERSITY OF
CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 1 June, 2006 9 to 12

PAPER 48

QUANTUM FIELD THEORY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a free field theory with scalar fields $\phi^k (k = 1, 2)$ and Lagrangian density

$$\mathcal{L}_{\text{free}} = \frac{1}{2}\partial_\mu \phi^1 \partial^\mu \phi^1 + \frac{1}{2}\partial_\mu \phi^2 \partial^\mu \phi^2 - \frac{1}{2}m^2\left((\phi^1)^2 + (\phi^2)^2\right).$$

Show that there is an internal symmetry generated by the infinitesimal variations

$$\phi^1 \to \phi^1 + \alpha\phi^2, \quad \phi^2 \to \phi^2 - \alpha\phi^1.$$

Using Noether's theorem, or otherwise, find the conserved charge Q associated with this symmetry.

Assuming now that the field theory is canonically quantized, express Q in terms of the fields ϕ^k and the conjugate momenta π^k. Using the expansions

$$\phi^k(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{p}}}} \left(a^k_{\boldsymbol{p}} + a^{k\dagger}_{-\boldsymbol{p}}\right) e^{i\boldsymbol{p}\cdot\boldsymbol{x}},$$

$$\pi^k(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{E_{\boldsymbol{p}}}{2}} \left(a^k_{\boldsymbol{p}} - a^{k\dagger}_{-\boldsymbol{p}}\right) e^{i\boldsymbol{p}\cdot\boldsymbol{x}},$$

show that

$$Q = -i\int \frac{d^3p}{(2\pi)^3} \left(a^{2\dagger}_{\boldsymbol{p}} a^1_{\boldsymbol{p}} - a^{1\dagger}_{\boldsymbol{p}} a^2_{\boldsymbol{p}}\right).$$

Find a one-particle state that is an eigenstate of Q, and determine the eigenvalue.

Suppose the interaction terms

$$\mathcal{L}_{\text{int}} = -\lambda\left(\phi^1\right)^4 - 2\mu\left(\phi^1\right)^2\left(\phi^2\right)^2 - \lambda\left(\phi^2\right)^4$$

are added to $\mathcal{L}_{\text{free}}$. What inequalities must λ and μ satisfy for there still to be a stable vacuum? For what values of λ and μ is Q still a conserved charge?

2 State the Feynman rules for particle scattering amplitudes in scalar ϕ^4 field theory, where the particles have rest mass m and the coupling constant is λ. Explain in outline how these rules are derived.

Draw a tree diagram (a diagram without loops) and a 1-loop diagram that contribute to the process where two incoming particles of 4-momenta p_1 and p_2 collide, and four outgoing particles of 4-momenta q_1, q_2, q_3 and q_4 are produced. What are the contributions to the scattering amplitude of these diagrams?

Suppose that one of the incoming particles has 3-momentum $\boldsymbol{p}$ and the other is at rest. What condition must $\boldsymbol{p}$ satisfy in order for it to be possible to have four outgoing particles?

3 Let Φ and ϕ be scalar Klein-Gordon fields, with Φ more than twice as massive as ϕ, and let the particles associated with these fields also be denoted by Φ and ϕ. Let ψ be a Dirac field of mass m.

Suppose the interaction terms in the Lagrangian density are

$$-G\overline{\psi}\psi\Phi - g\overline{\psi}\psi\phi$$

where G and g are real coupling constants. By considering the appropriate loop diagram, and Feynman rules, find the decay amplitude for $\Phi \to \phi + \phi$ to lowest order in G and g. Simplify as far as possible the traces of Dirac matrices that occur in your amplitude.

Suppose now that Φ is a pseudoscalar, rather than a scalar, and that the interaction terms are

$$iG\overline{\psi}\gamma^5\psi\Phi - g\overline{\psi}\psi\phi.$$

Find the field equation satisfied by ψ. By combining this with the equation satisfied by $\overline{\psi}$, determine the expression for the 4-divergence of the axial current $\overline{\psi}\gamma^\mu\gamma^5\psi$.

4 Write an essay on: Gauge invariance and its consequences in quantum electrodynamics.

END OF PAPER