UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 6 June 2006 9 to 11

PAPER 43

ACTUARIAL STATISTICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider the total amount of the claims arising from traffic accidents for which an insurance company receives at least one claim. Let N be the number of claims from one such accident and let the claim sizes $X_1, X_2, \ldots$ be independent identically distributed random variables, independent of N. Let $p_n = \mathbb{P}(N = n)$, so that $p_0 = 0$, and assume that

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1}, \quad n = 2, 3, \ldots,$$

where $a, b \in \mathbb{R}$ are known.

Suppose that the claim sizes are discrete with $f_k = \mathbb{P}(X_1 = k)$, $k = 1, 2, \ldots$, where $\sum_{k=1}^{\infty} f_k = 1$, and assume that the p_n's and the f_k's are known. Let $g_k = \mathbb{P}(X_1 + \ldots + X_N = k)$, $k = 1, 2, \ldots$.

By considering probability generating functions, derive a recursion formula for the g_k's in terms of known quantities.

Write down the recursion if

$$p_n = \frac{e^{-\lambda}\lambda^n}{(1 - e^{-\lambda})n!} \quad n = 1, 2, \ldots$$

2 A portfolio consists of n independent risks. For the i^{th} risk, the number of claims in a year has a Poisson distribution with parameter λ_i and the claims are independent exponentially distributed random variables with mean μ, independent of the number of claims. Let S_i be the total amount claimed in a year for risk i. Find the moment generating function of S_i, and show that $S = S_1 + \ldots + S_n$ has a compound Poisson distribution.

Now suppose that $\lambda_1, \ldots, \lambda_n$ are independent identically distributed random variables with density

$$f(\lambda) = \frac{\alpha^m \lambda^{m-1} e^{-\alpha\lambda}}{(m-1)!}, \quad \lambda > 0$$

for $\alpha > 0$ and $m \in \mathbb{N}$, so that the number of claims for each risk in one year has a mixed Poisson distribution with mixing density $f(\lambda)$. Find the distribution of the total number of claims on the whole portfolio in one year.

Show that the total amount S claimed in one year on the whole portolio has a compound mixed Poisson distribution, and identify the mixing distribution for the Poisson parameter.

3 Explain what is meant by a classical risk model with positive premium loading factor.

Assume that the adjustment coefficient is the unique positive solution of

$$M(r) - 1 = (1+\theta)\mu r,$$

where $M(r)$ is the claim size moment generating function, μ is the mean claim size and θ is the premium loading factor. State and prove Lundberg's inequality for the probability of ruin.

Find the adjustment coefficient R when claims are exponentially distributed with mean μ.

Determine whether R is greater or smaller than the adjustment coefficient R_μ for claims that are exactly μ, and comment briefly on the corresponding Lundberg bounds.

4 Let Y_i be the number of claims on a group life insurance policy covering m_i lives in year i, $i = 1, \ldots, n$. Suppose that

$$\mathbb{P}(Y_i = x) = \binom{m_i}{x} \theta^x (1-\theta)^{m_i - x}, \quad x = 0, \ldots, m_i,$$

where $\theta \in (0,1)$ has prior density $f(\theta)$. Let $X_i = Y_i/m_i$ and assume that, given θ, $X_1, \ldots, X_n$ are conditionally independent. Suppose θ is estimated by $\hat{\theta} = a_0 + \sum_{i=1}^n a_i X_i$ where $a_0, a_1, \ldots, a_n$ are chosen such that $\mathbb{E}_{x,\theta}[(\theta - \hat{\theta})^2]$ is minimised. Show that $\hat{\theta}$ can be written in the form

$$\hat{\theta} = Z \frac{\sum_{i=1}^n m_i X_i}{\sum_{i=1}^n m_i} \; + \; (1-Z)\mathbb{E}[\theta]$$

where you should specify Z.

Now suppose that $f(\theta) = 1$ for $0 < \theta < 1$ and that $n = 2$. Find $\hat{\theta}$, and compare it with the Bayesian estimate of θ with respect to quadratic loss.

END OF PAPER