UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 9 June, 2006 1.30 to 3.30

PAPER 40

OPTIMAL INVESTMENT

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) In a discrete-time model, there is a single productive asset (a tree) which in period t produces random output (fruit) d_t. Shares in the tree are traded in period t at price S_t. An agent enters period t holding θ_t shares in the tree. He receives the fruit due to his shares, then changes his holding of shares to θ_{t+1}, and consumes C_t, the fruit he currently holds, before passing to the next time period. Write down the budget equation linking θ_t, θ_{t+1} and C_t. The agent has objective

$$E\sum_{t\geqslant 0}\beta^t U(C_t),$$

where U is concave and increasing. Develop the Lagrangian theory of this problem to explain what the state-price density process is, and how it is related to the other variables of the problem.

(ii) Explain how the model of part (i) can be used to study an equilibrium involving many agents, identifying clearly what are the primitives of the problem, and what are the derived quantities. Illustrate your answer by considering the situation where there are J agents, agent j having objective

$$E\sum_{t\geqslant 0} -\beta_j^t\gamma_j^{-1}\exp(-\gamma_j C_t)$$

where $0 < \beta_j < 1$, $0 < \gamma_j$.

2 An investor invests in a riskless bank account and a single risky asset. If S_t is the price of the stock at time t, then S is modelled as

$$dS_t = S_t(\sigma_t dW_t + \mu_t dt),$$

where W is a standard Brownian motion. Let r_t denote the riskless rate of interest at time t. Write down and briefly explain the equation which governs the evolution of the wealth of the investor.

Assuming that the processes r, μ, σ and σ^{-1} are all bounded, determine as explicitly as you can the form of the wealth process w and the portfolio process θ that would be used if the objective of the agent is to maximise $E[U(w_T)]$, where U is increasing and concave, and $T > 0$ is fixed.

3 An agent invests in a riskless bank account yielding constant interest rate r, and in a stock whose dynamics are given by

$$dS_t = S_t\sigma(dW_t + \alpha dt),$$

where W is a standard Brownian motion, $\sigma > 0$ is constant, and α is constant but not known to the agent. The agent has a prior $N(\hat{\alpha}_0, \tau_0^{-1})$ distribution for α, and has to estimate the true value from observing the stock. Argue that at later time t the agent's posterior distribution for α is still Gaussian, with variance $\tau_t^{-1} \equiv (t + \tau_0)^{-1}$ and with mean $\hat{\alpha}_t$ which satisfies a stochastic differential equation

$$d\hat{\alpha}_t = \tau_t^{-1} d\hat{W}_t,$$

where $\hat{W}$ is a Brownian motion in the filtration of S.

The agent's objective is to maximise $E\log(w_T)$, where $T > 0$ is fixed and w_t is his wealth at time t. Express the optimal wealth process in terms of the state-price density process, and deduce that the agent must optimally invest a proportion

$$\frac{\sigma\hat{\alpha}_t - r}{\sigma^2}$$

of his wealth at time t in the risky asset.

4 Consider a market with a fixed riskless rate of interest $r \geqslant 0$, and n stocks, whose price processes obey

$$dS_t^i/S_t^i = \sum_{j=1}^{n} \sigma_{ij} dW_t^j + \mu^i dt, \quad (i = 1, \ldots, n)$$

where W is a Brownian motion in $\mathbb{R}^n$, and σ is $n \times n$ non-singular. What is the *state-price density* process ζ for this market, and what is its significance?

The geometric market index J is defined to be

$$J_t \equiv \left\{ \prod_{i=1}^{n} S_t^i \right\}^{1/n}.$$

Prove that

$$\log(J_t/J_0) = n^{-1}\left[\, \mathbf{1} \cdot \sigma W_t + \{\mathbf{1} \cdot \mu - \tfrac{1}{2}\mathrm{tr}(V)\}\, t \,\right]$$

where $\mathbf{1} = (1, \ldots, 1)^T$, and $V \equiv \sigma\sigma^T$.

An agent with initial wealth w_0 invests in this market, choosing a portfolio process θ. Write down the equation for the evolution of his wealth. His aim is to invest in such a way as to do well relative to the index; his objective is $E[\,U(w_T/J_T)\,]$, where U is the CRRA utility $U(x) = x^{1-R}/(1-R)$, for some positive $R \neq 1$. Explain why his optimally-invested wealth process w^* must satisfy

$$\frac{1}{J_T}\, U'\left(\frac{w_T^*}{J_T}\right) = \lambda \zeta_T$$

for some $\lambda > 0$ (a rigorous argument is not required.) How is the constant λ to be determined?

Find the optimal solution, and show that the agent should optimally split his wealth among the stocks in fixed proportions, given by the vector

$$R^{-1}V^{-1}(\mu - r\mathbf{1}) + (1 - R^{-1})n^{-1}\mathbf{1}.$$

END OF PAPER