UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 7 June, 2006 1.30 to 4.30

PAPER 4

REPRESENTATION THEORY OF SYMMETRIC GROUPS

*Attempt **THREE** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

Throughout, for $n \in \mathbb{N}$, $G = \Sigma_n$ is the symmetric group of degree n and F is a field of characteristic p.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 What is a partition of n? Explain why the partitions of n index the ordinary irreducible representations of G. Construct an irreducible rational representation of G for each partition of n (you should define all terms you use), and prove that the representations you obtain are in fact absolutely irreducible.

State and prove the corresponding classification result over F.

2 For partitions λ, μ of n, define what is meant by standard and semi-standard λ-tableaux. Define the dominance ordering on partitions and also $K_{\lambda,\mu}$, the (λ, μ)th Kostka number. Define the permutation module M^{μ} of G corresponding to the partition μ of n.

Show that all the composition factors of M^{μ} have the form D^{λ} with λ strictly dominating μ, except when μ is p-regular, when D^{μ} occurs precisely once. State a corresponding result for the composition factors of the Specht module S^{μ}. Quoting any results you need about the standard basis of the Specht module, describe the general shape of the p-modular decomposition matrix.

State Young's Rule which describes these factors, and use this Rule to show that, for any λ, $K_{\lambda,\lambda} = K_{(n),\lambda} = 1$ and that $K_{\lambda,(1^n)} = f^{\lambda}$, where $f^{\lambda} = \chi^{\lambda}(1)$ is the number of standard tableaux of shape λ. Show also, in the usual notation, that if $m \leqslant n/2$

$$[n-m][m] = [n] + [n-1,1] + [n-2,2] + \cdots + [n-m,m],$$

and hence deduce an expression for $\dim S^{(n-m,m)}$.

3 Describe the conjugacy classes of A_n. Hence classify the ordinary irreducible representations of A_n. Classify the 1-dimensional representations of A_n and, when $n \neq 5$ find the lowest dimension ($\neq 1$) of an ordinary irreducible representation of A_n. For $2 \leqslant n \neq 3, 6$, show that up to isomorphism there is a unique ordinary irreducible representation of A_n which is of dimension $n-1$.

4 Define a pair of partitions $(\mu^{\sharp}, \mu)$ and the set of sequences $s(\mu^{\sharp}, \mu)$. If T is a λ-tableau, define the generalised polytabloid $e_T^{\mu^{\sharp},\mu}$ and the generalised Specht module $S^{\mu^{\sharp},\mu}$. Assuming the Basic Combinatorial Theorem (or its equivalent formulations), prove that $S^{\mu^{\sharp},\mu}$ has a Specht series, with factors given by $[0]^{[\mu^{\sharp},\mu]\cdot}$. Deduce the Littlewood-Richardson Rule giving the constituents of $[\mu]^{[\lambda]\cdot}$.

Find a Specht series for $S^{(4,2^2,1)} \uparrow^{\Sigma_{10}}$.

5 Prove the Determinantal Form. State the Murnaghan-Nakayama Rule and briefly sketch a proof. Illustrate the Rule by evaluating the ordinary character $\chi^{(4^2,3)}$ on the conjugacy class defined by partition $(5,4,2)$

Explain how the Branching Rule can be considered a special case of the Murnaghan-Nakayama Rule.

What is the p-weight of a partition? Suppose that λ has p-weight w. Define the element $\rho = (1,\ldots,p)(p+1,\ldots,2p)\cdots((w-1)p+1,\ldots,wp) \in G$ and let $\pi \in \Sigma_{n-wp}$. Evaluate $\chi^{\lambda}(\pi\rho)$.

6 Given a partition define a hook and a rim hook. What is the hook graph? Describe the abacus notation for any given partition. Define β numbers and describe their relationship to first column hook lengths. Define the p-core of a partition and show that it is well-defined.

What is a p-block B of FG and what does it mean to say that an indecomposable module belongs to B? State the "Nakayama Conjecture" on the p-block structure of FG and write an essay indicating the main steps in its proof.

END OF PAPER