UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 12 June, 2006 1.30 to 3.30

# PAPER 36

# SPREAD OF EPIDEMICS AND RUMOURS

*Attempt* ***THREE*** *questions.*

*There are* ***FOUR*** *questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** The population consists of only susceptibles and infectives. A susceptible that is infected will stay infectious indefinitely.

We suppose that the population is closed, i.e. $X(t) + Y(t) = n$ for all $t \geqslant 0$ where $X(t)$ and $Y(t)$ denote the number of susceptibles and infectives at time $t$, with the initial condition $X(0) = an$ and $Y(0) = (1-a)n$, $a \in (0,1)$. At time $t$, a susceptible individual turns infected at rate $\lambda Y(t)/n$ where $\lambda > 0$.

(a) Let $X_n(t) = X(t)/n$ be the fraction of susceptible individuals at time $t$. Show that

$$X_n(t) \;=\; a - \int_0^t \lambda X_n(s)[1 - X_n(s)]\, ds - \epsilon_n(t)\,,$$

where $\epsilon_n(t)$ is expressed in terms of a standard Poisson process.

(b) Prove that

$$\lim_{n\to\infty} \sup_{0\leqslant t\leqslant T} |X_n(t) - x(t)| \;=\; 0\,, \quad \text{a.s.}$$

where

$$x(t) \;=\; \frac{ae^{-\lambda t}}{1 - a(1 - e^{-\lambda t})}\,.$$

*[Hints: You may assume without proof the following results:*

- Let $\mathcal{N}$ be a standard Poisson process and $\epsilon$ a positive constant. Then for $T > 0$,

$$\mathbb{P}\Big(\sup_{0\leqslant t\leqslant T} |\mathcal{N}(t) - t| \geqslant T\epsilon\Big) \leqslant 2e^{-Th(\epsilon)}\,,$$

where $h(t) = (1+t)\log(1+t) - t$.

- Gronwall's lemma: Let $f$ be a real-valued function satisfying

$$f(t) \leqslant \alpha + \beta \int_0^t f(s)ds\,, t \geqslant 0\,.$$

*Then $f(t) \leqslant \alpha e^{\beta t}$, for $t \geqslant 0$.* ]

**2** Fix $p \in (0,1)$ and let $G(n,p)$ be the Erdös-Rényi random graph with $n$ nodes, where each pair of nodes is connected independently with probability $p$ (here $p$ is constant and does not depend on $n$).

(a) Show that

$$\mathbb{P}\left(G(n,p) \text{ is disconnected}\right) \leqslant \sum_{s=1}^{[n/2]} \left(n(1-p)^{n/2}\right)^s ,$$

and deduce that

$$\lim_{n\to\infty} \mathbb{P}\left(G(n,p) \text{ is connected}\right) = 1 .$$

[*Hint: Find an upper bound on the probability that the graph admits an isolated component of size $s$, for $s \in \{1,\ldots,[n/2]\}$.* ]

(b) In the following, we are interested in the diameter of $G(n,p)$ for $p \in (0,1)$.

(i) For $X$ a random variable in $\{0,1,2,\ldots\}$ show that

$$\mathbb{P}\left(X > 0\right) \leqslant \mathbb{E}(X) .$$

(ii) Let $X$ be the number of pairs of nodes with no common neighbour. Show that

$$\mathbb{E}(X) = \binom{n}{2}(1-p^2)^{n-2} .$$

(iii) Deduce that

$$\lim_{n\to\infty} \mathbb{P}\left(\text{Diameter of } G(n,p) \leqslant 2\right) = 1 .$$

**3** We consider an Erdös-Renyi random graph $G(n,p)$ with $n$ nodes, where each pair of nodes is connected independently with probability $p$. Given $i,j,k$ three distinct nodes in $G(n,p)$, we say that the graph contains the triangle $(i,j,k)$ if the links $(i,j)$, $(j,k)$ and $(i,k)$ are present in $G(n,p)$. Let $X$ be the number of triangles in $G(n,p)$.

(a) Prove that if

$$\lim_{n\to\infty} np = 0\,,$$

then the number of triangles in $G(n,p)$ is zero with probability tending to 1.

[*Hint: Use the following inequality:* $\mathbb{P}(X>0) \leqslant \mathbb{E}(X)$. ]

(b) Assume that

$$\lim_{n\to\infty} np \;=\; \infty\,.$$

(i) Note that

$$X \;=\; \sum X_{ijk}\,,$$

where $X_{ijk}=1$ if the triangle $ijk$ is present in $G(n,p)$ and $X_{ijk}=0$ otherwise, and where the sum is over distinct nodes $i$, $j$ and $k$.

Show that

$$\mathbb{E}(X^2) \;=\; \sum_{l=0}^{3}\binom{n}{3}\binom{3}{l}\binom{n-3}{3-l}p^{2\binom{3}{2}-\binom{l}{2}}\,.$$

(ii) Using Chebyshev's inequality, establish the following inequality

$$\mathbb{P}(X=0) \leqslant \frac{\mathrm{Var}(X)}{(\mathbb{E}X)^2}\,,$$

where $\mathrm{Var}(X)$ is the variance of $X$.

(iii) Prove that

$$\lim_{n\to\infty}\mathbb{P}\left(\text{there is at least one triangle in } G(n,p)\right) \;=\; 1\,.$$

**4** We consider an undirected graph $G = (V, E)$ where $V = \{1, \ldots, n\}$ is the set of nodes and $E \subset V \times V$ is the set of edges and let $A$ be its adjacency matrix, i.e. $A_{ij} = A_{ji} = 1$ if $i$ and $j$ are connected in $G$ (i.e. $(i, j) \in E$) and 0 otherwise. We assume that each node can be in one of three states: susceptible, infected or removed, and that initially there is at least one infected node.

The evolution of the epidemic is described by the following discrete-time model: an infected node stays infected for a unit of time during which it infects each of its neighbours with probability $\beta$ and is then removed.

(a) Let $X_i(k)$ be the indicator that node $i$ is infected at time $k$, and $Y_i(k)$ the indicator that node $i$ is removed at time $k$. Show that

$$\mathbb{P}\left(Y_i(\infty) = 1\right) \;=\; \mathbb{P}\left(\text{node } i \text{ ever gets infected}\right) \leqslant \sum_{k=0}^{\infty} \sum_{j=1}^{n} \beta^k (A^k)_{ij} X_j(0)\,.$$

(b) Deduce that

$$\mathbb{E}\left[\sum_{i=1}^{n} Y_i(\infty)\right] \leqslant \sum_{k \geqslant 0} (1, \ldots, 1)(\beta A)^k X(0)\,,$$

where

$$X(0) = \begin{pmatrix} X_1(0) \\ \vdots \\ X_n(0) \end{pmatrix}.$$

(c) Let $\rho$ be the spectral radius of the adjacency matrix $A$ (the eigenvalue of $A$ with the largest absolute value) and suppose that $\beta\rho < 1$. Show that

$$\mathbb{E}\left[\sum_{i=1}^{n} Y_i(\infty)\right] \leqslant \frac{1}{1 - \beta\rho} \sqrt{n \sum_{i=1}^{n} X_i(0)}\,.$$

Deduce a sufficient condition for a small outbreak.

## END OF PAPER