UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 9 June, 2006 1.30 to 3.30

PAPER 3

TOPICS IN INFINITE GROUPS

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let F_n be the free group of rank n for $n \geqslant 2$ and H a finite index subgroup of F_n. Prove that H is a free group.
(You may assume that the free group of rank k is the free product of k copies of $\mathbb{Z}$.)

If H has index i then state and prove a formula for the rank of H.

Suppose now that H_j is a sequence of subgroups of F_n with the index $i_j = [F_n : H_j]$ tending to infinity with j. What can you say about the ratio of the rank of H_j to its index in F_n as $j \to \infty$?
Prove that if θ is a homomorphism from F_n to itself then either θ is a surjection or the image has infinite index in F_n.

2 Starting from the fact that every group is a quotient of a free group, define what it means for a group G to be finitely presented. Prove that if we take any finite generating set for G then this notion is independent of our choice.

Now suppose that a group G has subgroups A and B with ϕ an isomorphism from A to B. Define the HNN extension $G*_\phi$ with stable letter t and what is meant by a normal form. Prove that every element of $G*_\phi$ has a unique normal form.

Define a reduced sequence and deduce Britton's Lemma: a reduced sequence containing an appearance of t or t^{-1} is not the identity in $G*_\phi$.

Let $F_2 = \langle a, b\rangle$ be the free group of rank 2 and let $H \leqslant F_2$ be a subgroup which is free of infinite rank on $\{w_1, w_2, w_3, \ldots\}$ for $w_n \in F_2$.
Prove that the HNN extension of F_2 formed by taking the identity isomorphism from H to itself is finitely generated but not finitely presented.

(You may assume standard facts about free products.)

3 (a) Show that the property of being finitely generated is preserved by quotients, extensions and subgroups of finite index.
(You may assume the universal property of free groups and the Nielsen-Schreier index formula.)

Recall that G is locally finite if every finitely generated subgroup of G is finite.
Give an example with justification of a locally finite group that is not finite.
Prove that the property of being locally finite is preserved by subgroups, quotients and extensions.

(b) For this part you may assume the existence of an infinite group G and a large prime p such that every proper non-trivial subgroup of G has order p.
(i) Show that G is finitely generated.
(ii) Show that G is simple.

If we replace the above condition on the subgroups of G with: every element of G has order p then do (i) or (ii) still hold?

4 If H is a subgroup of G then define the left regular representation of G on the left cosets of H in G.
Prove that if H has finite index in G then there is N normal in G with N having finite index in H.

Describe Higman's construction of an infinite finitely presented group with no proper finite index subgroups. (You may assume that the respective component groups embed in HNN extensions and in free products with amalgamation.)

Define a maximal normal subgroup of the group G and prove that if G is finitely generated then every proper normal subgroup of G is contained in a maximal normal subgroup.
Explain how we can deduce the existence of an infinite finitely generated simple group.
If G is finitely generated then is every proper subgroup of G contained in a maximal normal subgroup?

5 Let $\theta : G \to H$ be a surjective homomorphism with kernel K. Show that there is a natural bijection between subgroups of H and subgroups of G containing K.
If B is a subgroup of H with index i then show that $\theta^{-1}(B)$ has index i in G.
Suppose that A has index i in G.
(i) Does $\theta(A)$ have index i in H? What if A contains K?
(ii) Taking $g \in G$, does gAg^{-1} have index i in G?

Define the abelianisation $\overline{G}$ of a group G and show that if N is normal in G with G/N abelian then the natural projection $\pi : G \to G/N$ factors through $\overline{G}$.

Now suppose that S is a finitely generated group with exactly one subgroup of index i for each $i \in \mathbb{N}$.
By considering large primes p or otherwise, show that the abelianisation $\overline{S}$ possesses a surjective homomorphism to $\mathbb{Z}$.
Show that in fact $\overline{S} = \mathbb{Z}$.

Define what it means for a group to be residually finite.
If furthermore S is residually finite then deduce that $S = \mathbb{Z}$.

END OF PAPER