UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 8 June, 2006 1.30 to 3.30

PAPER 28

ANALYTIC NUMBER THEORY

*Attempt **TWO** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Illustrate the method of Tchebychev by proving that $\pi(x) = O(x/\log x)$. Hence establish the result of Mertens that

$$\sum_{p \leqslant x} \frac{\log p}{p} = \log x + O(1).$$

Deduce by partial summation, or otherwise, that, for any $\delta > 0$, the series $\sum 1/(p(\log p)^\delta)$, summed over all primes p, converges.

2 Prove that $\zeta(s) \neq 0$ on the line $\sigma = 1$.

Write down a relation between

$$\int_0^x \psi(u) du$$

and $\zeta'(s)/\zeta(s)$, where ψ is the Tchebychev function. Describe how it enables one to verify that the integral is asymptotic to $\frac{1}{2}x^2$ as $x \to \infty$ and so gives the prime number theorem.

3 State and prove the functional equation for $\zeta(s)$.

State also the Riemann-von Mangoldt formula. Deduce from the latter that if $\gamma_1, \gamma_2, \ldots$ is the increasing sequence of positive ordinates of the zeros of $\zeta(s)$ then $\gamma_n \sim 2\pi n/\log n$ as $n \to \infty$. Explain what is meant by the Riemann hypothesis in this context.

4 Describe the main ideas of the Selberg upper-bound sieve.

Indicate how it leads to the result that the number of primes p with $p \leqslant N$ such that $p + 2$ is prime is $\ll N/(\log N)^2$.

5 Prove Dirichlet's theorem on primes in arithmetical progressions assuming that $L(1, \chi) \neq 0$ for a real, non-principal character χ.

It is known from the Siegel-Walfisz theorem that the number $\pi(x, q, a)$ of primes $p \leqslant x$ in the arithmetical progression a, $a + q$, $a + 2q$, $\ldots$ with $(a, q) = 1$ satisfies $\pi(x, q, a) \sim (1/\phi(q))x/\log x$ as $x \to \infty$. Deduce by the partial summation formula, or otherwise, that

$$\sum_{p \leqslant x,\, p \equiv a (\mathrm{mod}\, q)} \frac{1}{p} \sim \frac{1}{\phi(q)} \log\log x \quad \text{as} \quad x \to \infty.$$

END OF PAPER