UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Monday 12 June, 2006   9 to 12

PAPER 25

SET THEORY

*Attempt **FOUR** questions.*

*There are **ELEVEN** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write an essay on computable functions $\mathbb{N} \rightarrow \mathbb{N}$. You should include *inter alia* Rice's theorem and the theorem of Jockusch's that there is a recursive partition of $[\mathbb{N}]^3$ with no recursive monochromatic set.

2 Explain the device of Rieger-Bernays permutation models, and use it to prove the independence of the axiom of foundation from ZF. Extend your technique to prove the independence of the axiom of choice from ZF minus foundation.

3 Prove Kruskal's theorem on wellquasiordering of trees and deduce Friedman's Finite Form of it.

4 What is AD, the axiom of determinacy? Which games can you prove to be determinate? Establish that AD is inconsistent with AC.

5 Prove the independence of each of the following axioms from the remaining axioms of ZF: sumset, power set, replacement, extensionality, and infinity.

6 Show how a countable model $\mathfrak{M}$ of ZFC can be extended to a countable model $\mathfrak{M}'$ in which the ω_1 of $\mathfrak{M}$ has become countable-in-$\mathfrak{M}'$.

7 Write an essay on ultraproducts. In addition to Łoś's theorem you should cover the existence of saturated models, and give a direct proof of the compactness theorem for first-order logic.

8 Use Ramsey's theorem to prove (i) the Ehrenfeucht-Mostowski theorem, and (ii) the consistency of simple typed set theory with typical ambiguity and urelements.

9 What is a measurable cardinal? Explain the connection with elementary embeddings. Why is the embedding into the transitive collapse of the ultrapower not the identity? What can you say about the first ordinal moved by it?

10 State and prove the Erdös-Rado theorem on partitions with uncountable monochromatic sets. One consequence of this theorem is that a certain increasing function on ordinals is total. Give a condition (the *tree property*) for a supremum of iterates of this function to be a fixed point for it. Prove that any cardinal with the tree property must be strongly inaccessible.

11 Answer two of the following.

(i) Give a proof of the Ehrenfeucht-Mostowski theorem using ultraproducts not Ramsey's theorem.

(ii) Give a definition of $\mathbb{N}$ that does not involve quantification over infinite sets. Prove that it is the same as the usual definition.

(iii) What is an inner model? Why can there be no proof of the independence of AC using them?

END OF PAPER