UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 8 June, 2006 9 to 12

PAPER 24

CATEGORY THEORY

*Attempt **ONE** question from Section I and **TWO** from Section II.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION I

1

(i) State and prove some form of the Adjoint Functor Theorem.

(ii) Give, without detailed justification, an example of a functor between complete categories which preserves all (small) limits but fails to have a left adjoint. Which of the hypotheses of the Adjoint Functor Theorem (in the version you proved) fails in your example?

(iii) Use the Adjoint Functor Theorem to prove that the inclusion of the full subcategory of compact Hausdorff spaces in that of all topological spaces has a left adjoint. [You may assume Uryson's Lemma and Tychonoff's Theorem; if using the General Adjoint Functor Theorem, you may also assume the result that if X is a dense subspace of a Hausdorff space Y, then the cardinality of Y is at most $2^{2^{\text{card } X}}$.]

2 P. Freyd once claimed that 'the purpose of category theory is to show that which is trivial is trivially trivial'. Write a short essay arguing either for or against this assertion, illustrating your argument with results drawn from the course.

SECTION II

3 State the Yoneda Lemma for covariant functors $\mathcal{C} \to \mathbf{Set}$, where $\mathcal{C}$ is a locally small category.

Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor, and let $\mathcal{F}$ denote the arrow category $(1 \downarrow F)$, where 1 denotes a one-element set. Show that the slice category $[\mathcal{C}, \mathbf{Set}]/F$ is equivalent to $[\mathcal{F}, \mathbf{Set}]$. Show further that if $\mathcal{C}$ is small then F may be expressed as the colimit in $[\mathcal{C}, \mathbf{Set}]$ of a diagram of shape $\mathcal{F}^{\text{op}}$ whose vertices are representable functors.

Explain what is meant by a *cartesian closed category.* Show that if $\mathcal{C}$ is small then $[\mathcal{C}, \mathbf{Set}]$ is cartesian closed, and deduce that it is locally cartesian closed (i.e., that all its slice categories are cartesian closed). [The Adjoint Functor Theorem may *not* be assumed in this question.]

4 Explain what it means for an adjunction to be *idempotent*, and show that this condition is self-dual. Determine which of the following adjunctions are idempotent (in each case the left adjoint should be described explicitly):

(i) The functor **Cat** → **Set** which sends a small category to its set of morphisms, and its left adjoint.

(ii) The functor **Top** → **Top** which sends a space X to the same set with the indiscrete topology, and its left adjoint.

(iii) The functor from $\mathbf{Set}^{\mathrm{op}}$ to the category **Bool** of Boolean algebras which sends a set A to the Boolean algebra of all subsets of A, and its left adjoint.

5 Explain what is meant by the statement that an adjunction is *monadic*, and by the *monadic length* of an arbitrary adjunction. State the Precise Monadicity Theorem.

Let $\mathcal{C}_n$ denote the category whose objects are sets equipped with n partial unary operations $\omega_i : A \rightharpoonup A$ $(1 \leqslant i \leqslant n)$, such that $\omega_1(a)$ is defined for all a and, for each $i > 1$, $\omega_i(a)$ is defined iff ($\omega_{i-1}(a)$ is defined and) $\omega_{i-1}(a) = a$. Morphisms $A \to B$ in $\mathcal{C}_n$ are (total) functions f such that $\omega_i(f(a))$ (is defined and) equals $f(\omega_i(a))$ whenever $\omega_i(a)$ is defined, for all $i \leqslant n$. Show that the forgetful functor $\mathcal{C}_n \to \mathcal{C}_{n-1}$ (which 'forgets' the operation ω_n) is monadic. Show also that the composite adjunction between $\mathcal{C}_n$ and $\mathcal{C}_0 = \mathbf{Set}$ has monadic length n. [Hint: show that the monad on $\mathcal{C}_m$ induced by the forgetful functor $\mathcal{C}_n \to \mathcal{C}_m$ and its left adjoint, for any $n > m$, is independent of n.]

6 Define the notions of *monoidal category* and *symmetric monoidal category*. Give an example to show that a monoidal category may admit more than one symmetry. Prove the coherence theorem for monoidal categories. [You may assume the result that the 'α–λ–λ' and 'α–ρ–ρ' identities follow from the associativity pentagon and the 'α–λ–ρ identity'.]

END OF PAPER