UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 9 June, 2006 9 to 12

PAPER 23

SPECTRAL GEOMETRY

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Discuss the extent to which the geometric properties of a Riemannian manifold are spectrally determined, outlining, in particular, the historic development of the subject.

2 Show how the subspace $T \leqslant \mathbb{F}_3^4$ spanned by the vectors $(0,1,1,1)$ and $(1,1,-1,0)$ may be used to construct two lattices L^+ and L^- in $\mathbb{R}^4$ such that the flat tori $\mathbb{R}^4/L^\pm$ have the same length spectrum but are not isometric.

3 Prove that if $p : N \to M$ is a finite, normal, locally isometric covering of a Riemannian manifold with covering transformation group U, then the heat kernels K_N and K_M of N and M respectively are related by:

$$K_M(x,y,t) = \sum_{g \in U} K_N\left(\tilde{x}, g(\tilde{y}), t\right) \qquad (*)$$

where $\tilde{x}$ and $\tilde{y}$ are arbitrary, fixed choices of points such that $p(\tilde{x}) = x$ any $p(\tilde{y}) = y$.

[Your proof should establish that the right hand side of $(*)$ and any operation that you require are well-defined.]

Deduce Sunada's Theorem that, if N is a finite normal covering of M_1 and M_2 with covering transformation groups U_1 and U_2 that are Gassman equivalent in a group T of isometries of N, then M_1 and M_2 are isospectral.

[You may assume that, for any isometry h of N,

$$\int_N K_N\left(x, hgh^{-1}(x), t\right)\omega_N(x) = \int_N K_N(x, g(x), t)\omega_N(x). \quad]$$

4 Describe how to construct a Riemann surface M (of constant curvature -1) admitting a given group T of isometries. Describe the quotients $M_0 = T/M$ and, for a given subgroup U of T, $M_1 = U \setminus M$.

Explain how one can tell whether M_0 and M_1 admit metrics of constant curvature -1 without singularities. Derive the formulae for the Euler characteristics of M, M_0 and, when it has no singularities, M_1.

5 Certain groups T_i have subgroups U_i and V_i and generators A_i and B_i that have the permutation representations on the cosets of U_i and V_i given in the following table:

Group	Generators	Action on cosets of U_i	Action on cosets of V_i
T_1	A_1 B_1	(1 2 3)(4 5 6)(7 8 9)(10 11 12) (1 4 10)(2 7 6)(3 8 9)(5 11 12)	(1 2 3)(4 5 6)(7 8 9)(10 11 12) (1 4 10)(2 7 6)(5 8 9)(3 11 12)
T_2	A_2 B_2	(0 1 2 3 4 5 6) (0 1 4 6 2 5 3)	(0 1 2 3 4 5 6) (0 3 4 1 6 2 5)
T_3	A_3 B_3	(1) (0 3) (2 6 4 5) (0) (1 2 5) (3 6 4)	(4) (25) (0 1 6 3) (0) (1 4 3) (2 5 6)

Show how, using the relevant set of data, to construct isospectral surfaces of genus 2. Establish whether or not they are isometric when given a metric of constant curvature -1.

[You may assume any general results that you require without proof and may also assume, but should state clearly, any further specific hypotheses that you need.]

Indicate briefly how the data you have not used will produce similar pairs of surfaces of genus three and four and state, in each case, whether, when given a metric constant curvature -1, they are isometric.

END OF PAPER