UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 9 June, 2006 1.30 to 3.30

PAPER 22

SMOOTH FOUR-MANIFOLDS

*Attempt **ONE** question from Section A, and **ONE** from Section B.*

*There are **FIVE** questions in total.*

Section A questions carry 40% weight; and Section B questions carry 60% weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 (a) Show that the four-manifold $\mathbb{CP}^2 \# \mathbb{CP}^2$ contains a smoothly embedded copy of $\mathbb{RP}^3$, with disconnected complement.

[*You may find it useful to consider the unit tangent bundle of* S^2.]

(b) Let $X = \#_{i=1}^n \mathbb{CP}^2$. Show that the group of isometries of the intersection form Q_X contains a subgroup G isomorphic to the symmetric group S_n. Show also that every isometry in G is induced by a self-diffeomorphism of X.

[*An isometry is an invertible linear map* $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{Z})$ *which preserves the intersection form.*]

(c) Show that there is no continuous map $S^2 \times S^2 \to \mathbb{CP}^2 \# \mathbb{CP}^2$ of non-zero degree.

2 In this question, (X, g) is a closed, oriented, Riemannian four-manifold, $\Sigma \subset X$ a closed, oriented two-manifold smoothly embedded in X.

(a) Suppose that $L \to X$ is a U(1)-bundle and s a section with transverse zero-set $s^{-1}(0)$. Suppose that $s^{-1}(0) = \Sigma$ as oriented manifolds.

Explain why, if ∇ is a U(1)-connection in L, $i\,F(\nabla)$ may be regarded as an ordinary two-form (i.e. a section of Λ^2_X). Prove that $i\,F(\nabla)$ is closed, and that its cohomology class $[i\,F(\nabla)] \in H^2_{dR}(X)$ is independent of ∇.

(b) Prove that there is some U(1)-connection ∇ in L such that $i\,F(\nabla)$ is a g-harmonic two-form, and that when $b_1(X) = 0$, ∇ is unique up to gauge transformations.

(c) Suppose $X = \overline{\mathbb{CP}}^2$ (i.e. the complex projective plane with the opposite orientation from the usual complex orientation). Show that every line bundle L admits a unique gauge-orbit of ASD connections.

(d) Suppose $X = S^2 \times S^2$. Show that there is a line bundle L which does not admit any ASD connections. [*Consider* $L \oplus L^*$.]

3 (a) Prove that a flat SU(2)-connection in the trivial bundle over an open square $(-1,1)^2$ is equivalent to the trivial connection.

(b) State a relation between flat connections over a manifold and representations of its fundamental group.

Let T be the punctured 2-torus, $(S^1 \times S^1) \setminus \{\text{point}\}$. Let $E \to T$ be the trivial SU(2)-bundle, $\mathcal{G}$ its group of gauge transformations. Fix a point $x \in T$, and let

$$\mathcal{G}_x = \{u \in \mathcal{G} : u(x) = 1\}.$$

Let $\widetilde{M}(T)$ be the space of $\mathcal{G}_x$-orbits of flat SU(2)-connections in E. Explain how to identify $\widetilde{M}(T)$ with $S^3 \times S^3$.

(c) Let $M(T)$ be the space of $\mathcal{G}$-orbits of flat SU(2)-connections in E. Show that $M(T)$ has a dense open set U such that U is homeomorphic to $S^2 \times (0,1)$ and $M(T) \setminus U$ is homeomorphic to $[0,1] \cup [0,1]$ (disjoint union).

SECTION B

4 (a) Let E denote the trivial SU(2)-bundle over S^3, $\mathcal{A}_E$ the space of SU(2)-connections. The *Chern-Simons functional*

$$\mathrm{CS}\colon \mathcal{A}_E \to \mathbb{R}/\mathbb{Z}$$

is defined as follows: let W be a compact oriented four-manifold with boundary $\partial W = S^3$, $E_W \to W$ an SU(2)-bundle extending $E \to S^3$. Then

$$\mathrm{CS}(\nabla) = \frac{1}{8\pi^2} \int_W \mathrm{Tr}\, F(\widetilde{\nabla})^2 \mod \mathbb{Z},$$

where $\widetilde{\nabla}$ is an SU(2)-connection in W extending W.

Explain briefly why the value of $\mathrm{CS}(\nabla)$ is well-defined, independent of the choices of W, E_W and $\widetilde{\nabla}$. Show that

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{CS}(\nabla + ta) = \frac{1}{4\pi^2} \int_{S^3} \mathrm{Tr}\,(F(\nabla) \wedge a).$$

Show that if u is a gauge transformation of E which extends to a gauge transformation of E_W then

$$\mathrm{CS}(u \cdot \nabla) = \mathrm{CS}(\nabla).$$

(b) State and prove the Uhlenbeck-Donaldson compactness theorem for ASD connections in an SU(2)-bundle with Euler number 1 over a closed, oriented Riemannian four-manifold. [*State any analytic results you use.*]

5 In this question, X is a closed, oriented, simply connected, negative-definite four-manifold, g a Riemannian metric on X, and ∇ a g-ASD connection in an SU(2)-bundle E over X.

(a) Write down the three-term elliptic complex associated with ∇. Prove that it is a complex. (You do not need to prove that it is elliptic.) Write down a formula for the index of the associated elliptic operator.

(b) Suppose that ∇ is reducible, with holonomy group $S^1 \subset \mathrm{SU}(2)$. What is the dimension of H^0_∇? Justify your answer.

(c) State the Kuranishi model for the zero-set of a smooth map between Hilbert spaces which has Fredholm derivative.

Let M be the space of gauge-orbits of g-ASD connections. Suppose that ∇ has holonomy group S^1 and $H^2_\nabla = 0$. Show that there is an open neighbourhood of $[\nabla]$ in M which is homeomorphic to a neighbourhood of $[0]$ in

$$\mathbb{C}^d/S^1, \quad d = 4e(E) - 1,$$

where $\lambda \in S^1 \subset \mathbb{C}$ acts by scalar multiplication by λ^2.

END OF PAPER