UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 8 June, 2006 1.30 to 4.30

PAPER 21

COBORDISM

*Attempt **ALL** questions.*

*There are **FIVE** questions in total.*

Questions 1, 2, 3 and 4 carry equal weight.

*Question 5 carries **twice** the weight of any other question.*

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let L_1 and L_2 be compact C^∞-submanifolds without boundary of a C^∞-manifold M. Assume that L_1 and L_2 intersect transversally in M. Assume in addition that normal bundles $\nu(i_1)$ and $\nu(i_2)$ of embeddings $i_j : L_j \subset M$ are equipped with complex structures. Triples $[L_1, i_1, \nu(i_1)]$ and $[L_2, i_2, \nu(i_2)]$ represent elements of the complex cobordism group $\Omega_U^{**}(M, \emptyset)$. Describe geometrically the product

$$[L_1, i_1, \nu(i_1)] * [L_2, i_2, \nu(i_2)] \in \Omega_U^{**}(M, \emptyset)$$

and explain why it can be realised as a submanifold of M with a complex structure in the normal bundle of embedding.

2 Prove that the groups of unitary transformations $U(n) \subset Mat(n, \mathbb{C})$ of $\mathbb{C}^n$ and orthogonal transformations $O(n) \subset Mat(n, \mathbb{R})$ of $\mathbb{R}^n$ are C^∞-manifolds.

Assuming that the space of cosets G/H of a Lie subgroup H of a Lie group G is a C^∞-manifold with $G \to G/H$ smooth of everywhere maximal rank, find the set of homotopy equivalence classes of maps $[O(2n)/U(n), S^{n^2}]$, where $U(n)$ is considered as a Lie subgroup of $O(2n)$ under identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$?

3 Give the definition of the Thom class of a vector bundle ξ in the multiplicative cohomology theory $h^*(\cdot)$. Assume that a vector bundle ξ is oriented with respect to $h^*(\cdot)$ and let us fix a Thom class $u_h(\xi) \in h^{\dim \xi}(D(\xi), S(\xi))$. What is the Euler class $e_h(\xi)$? Show that if ξ admits a nowhere zero section then $e_h(\xi) = 0$.

4 What is the Whitney sum formula for Chern classes in complex cobordism of a sum $\eta \oplus \xi$ of two complex vector bundles?

Let $\mathbb{C}P(\eta)$ be a projectivisation of a complex vector bundle η over a base X, $\dim_{\mathbb{C}} \eta = n$. Define the one–dimensional tautological complex vector bundle $\eta(1)$ over $\mathbb{C}P(\eta)$.

Denote the projection $\mathbb{C}P(\eta) \to X$ by p. Using Whitney sum formula express Chern classes of an orthogonal complement $\eta(1)^\perp$ to $\eta(1)$ inside $p^*\eta$:

$$\eta(1) \oplus \eta(1)^\perp \equiv_{\mathbb{C}} p^*\eta,$$

in terms of Chern class $x = c_1(\eta(1))$ and pull–backs $p^*(c_k(\eta))$ of Chern classes of η, $k = 1, \ldots, n$.

Prove that $x = c_1(\eta)$ satisfies

$$x^n - p^*(c_1(\eta)) \cdot x^{n-1} + p^*(c_2(\eta)) \cdot x^{n-2} - \cdots + (-1)^n p^*(c_n(\eta)) = 0.$$

5 Define the Thom space of a vector bundle. Let $\mathbb{C}P^n \subset \mathbb{C}P^{n+m+1}$ be an embedding onto the first $n+1$ coordinates of $\mathbb{C}P^{n+m+1} = \{(z_0 : z_1 : \cdots : z_{m+n+1}\}$. Prove that

$$\mathbb{C}P^{n+m+1}/\mathbb{C}P^n$$

is homeomorphic to the Thom space $Th\left(\underbrace{\overline{\eta}_1 \oplus \cdots \oplus \overline{\eta}_1}_{n+1}\right)$ of the bundle $\underbrace{\overline{\eta}_1 \oplus \cdots \oplus \overline{\eta}_1}_{n+1}$ over $\mathbb{C}P^m$, where η_1 is the one-dimensional tautological complex vector bundle over $\mathbb{C}P^m$.

Let $\mathbb{C}P^k, \mathbb{C}P^m \subset \mathbb{C}P^n$ be coordinate embeddings as above $(k, m < n)$. Define complex structures ν_k, ν_m in the normal bundles of these embedding by means of the previous homeomorphism. Compute geometrically the product of two elements

$$[\mathbb{C}P^k, \nu_k] \in \Omega_U^{2(n-k)}(\mathbb{C}P^n, \emptyset), \quad [\mathbb{C}P^m, \nu_m] \in \Omega_U^{2(n-m)}(\mathbb{C}P^n, \emptyset).$$

END OF PAPER