UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 7 June, 2006 9 to 12

PAPER 20

SYMPLECTIC TOPOLOGY

*Attempt **THREE** questions.*

*There are **FIVE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the Lie derivative, and prove Cartan's formula. Deduce that the group of compactly supported symplectomorphisms of a connected symplectic manifold M acts transitively on points of M.

Let $S^2 \subset \mathbb{R}^3$ denote the unit sphere with its standard symplectic structure. Let $C = S^2 \cap \{z = 0\}$ and $C' = S^2 \cap \{z = \frac{1}{2}\}$ denote the circles given by intersecting the sphere with the given planes. Is there a symplectomorphism $\phi : S^2 \to S^2$ taking C to C'? Justify your answer.

2 State the symplectic neighbourhood theorem for symplectic submanifolds of a symplectic submanifold, and outline the proof using Moser's method.

Let C be a smooth surface of genus g. Prove that if symplectic 4-manifolds X and Y contain embedded symplectic submanifolds diffeomorphic to C and of self-intersection zero, then the fibre sum $X\#_{C\sim C'}Y$ carries a natural symplectic structure.

Show by example that if $C \subset X$ and $C' \subset Y$ are only smoothly embedded square zero surfaces of the same genus, the fibre sum $X\#_{C\sim C'}Y$ need not admit a symplectic structure.

3 What does it mean to blow up a symplectic manifold at a point $p \in X$? Prove that the blow-up of X at p carries a symplectic structure. Is this well-defined up to symplectomorphism?

State without proof a formula for the first Chern class of the blow-up.

Equip the four-torus T^4 with its standard symplectic structure. Show that a Lefschetz pencil of genus g curves on T^4 has exactly $2g - 2$ base points.

4 What does it mean for an almost complex structure J to be compatible with a symplectic form? Prove that every symplectic manifold (M, ω) has a connected non-empty space of compatible almost complex structures.

What does it mean for a compatible J to be regular for a J-holomorphic map $u : \mathbb{P}^1 \to M$? Suppose $\{\partial_s u + J(u)\partial_t u = 0\}$ in local co-ordinates and let ξ be a tangent vector to u. By differentiating this expression with respect to ξ, show that if u is a constant map then the linearisation D_u is a direct sum of copies of the Dolbeault operator $\overline{\partial} : \Omega^{0,0}(\mathbb{P}^1) \to \Omega^{0,1}(\mathbb{P}^1)$. Deduce every J is regular for all curves u representing the zero homology class.

Now suppose J is regular and $\dim_{\mathbb{R}}(M) = 4$. State a theorem giving the dimension of the space of J-holomorphic spheres representing a class $A \in H_2(M;\mathbb{Z})$. Deduce that if $[A] \cdot [A] \leqslant -2$ the space of J-holomorphic spheres in class A is empty.

5 What is a symplectic capacity on a symplectic manifold (X, ω) ?

Assuming the existence of a symplectic capacity c, prove the rigidity theorem: $\mathrm{Symp}(X)$ is C^0-closed in $\mathrm{Diff}(X)$, the group of all diffeomorphisms of X.

Now let c be a capacity on subsets of $\mathbb{R}^{2n}$. Let $U \subset \mathbb{R}^{2n}$ be an open non-empty bounded subset and $W \subset \mathbb{R}^{2n}$ a codimension 2 linear subspace. Write $U+W = \{u+w \mid u \in U,\, w \in W\}$. Prove $0 < c(U+W) < \infty$ if $W^{\perp}$ is not isotropic. [Hint: to show finiteness, embed $U+W$ in something standard.]

END OF PAPER