UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 2 June, 2006 9 to 12

PAPER 2

NOETHERIAN ALGEBRAS

Attempt ***TWO*** *questions from section A and* ***TWO*** *from section B.*

There are ***THREE*** *questions in each section.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Let R be a left Noetherian ring. Suppose that M is a left R-module and let N be a submodule of M. Show that M is Noetherian if and only if both N and M/N are Noetherian. Deduce that any finitely generated left R-module is Noetherian.

What is a *poly-(cyclic or finite)* group? Show that the group algebra RG of any poly-(cyclic or finite) group G is left Noetherian.

2 Define the *Jacobson radical* J of a ring R. Show that any proper right ideal of R is contained in a maximal right ideal. State and prove Nakayama's Lemma.

Let M be an Artinian right R-module and let $f : M \to M$ be an injective right R-module homomorphism. By considering images of successive powers of f or otherwise, show that f is surjective.

Suppose that R is *semilocal*, that is, R/J is right Artinian. Let V be a finitely generated right R-module and let $\alpha : V \to V$ be a right R-module homomorphism such that

$$\alpha^{-1}(VJ) \subseteq VJ.$$

Using Nakayama's Lemma or otherwise, show that α is surjective.

3 Let R be a ring. Define the terms *essential right ideal* and *regular element*.

What is the *classical right ring of quotients* of R? State Goldie's Theorem.

Suppose that R is a semiprime right Noetherian ring. Show that if a right ideal of R contains a regular element, then it is essential. Assuming that every essential right ideal of R contains a regular element, prove that the classical right ring of quotients of R exists and is semisimple Artinian.

[*You may assume Ore's Theorem, as well as results on Artinian rings.*]

SECTION B

4 What is a *filtration* on a ring R? Let $(F_nR)_{n\in\mathbb{Z}}$ be a filtration on R and let M be a filtered right R-module. Explain briefly how gr$M$ becomes a right module for the associated graded ring gr$R$ of R.

Let N be a submodule of M. Define the terms *subspace filtration* and *quotient filtration*, and show that there exists a short exact sequence of right gr$R$-modules

$$0 \to \text{ gr } N \to \text{ gr } M \to \text{ gr } (M/N) \to 0.$$

Suppose that the filtration on R is complete and negative. Show that $F_{-1}R$ is contained in the Jacobson radical of R. Assuming that gr$R$ is right Noetherian, show that R is also right Noetherian.

5 Let R be a left Noetherian ring and let I be an ideal of R. Define the *prime radical* N of R. What is a *minimal prime* over I? Show that there are only finitely many minimal primes over I, and that I contains a finite product of some of them. Deduce that N is nilpotent.

Suppose further that R is commutative and let M be a finitely generated R-module. What is an *associated prime* of M? Show that M has only finitely many associated primes.

Show that every minimal prime of R occurs as an associated prime of R viewed as a module over itself.

6 Let k be a field of characteristic zero, let R be an almost commutative k-algebra and let M be a finitely generated left R-module. Define the *dimension* $d(M)$ and *multiplicity* $m(M)$ of M, and explain briefly why these concepts are well-defined.

Suppose now that N is a submodule of M. Show that $d(M) = \max\{d(N), d(M/N)\}$ and also that if $d(N) = d(M/N)$, then $m(M) = m(N) + m(M/N)$.

State and prove Bernstein's inequality for finitely generated modules over the Weyl algebra $A_n(k)$, and use it to deduce that any finitely generated $A_n(k)$-module of dimension n is Artinian.

[*You may assume that the centre of $A_n(k)$ is k.*]

END OF PAPER