UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 6 June, 2006 1.30 to 4.30

PAPER 19

REPRESENTATIONS OF COMPACT LIE GROUPS

*Attempt **QUESTION 1** and **THREE** other questions.*
*There are **SIX** questions in total.*
Question 1 carries 30 marks and all other questions carry 20 marks each.

You may use without proof the following formulae, pertaining to the irreducible representation V_λ with highest weight λ of a compact, connected Lie group G:

1. *The Weyl character formula:*

$$\chi_\lambda = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha>0}(1-e^{-\alpha})}$$

2. *The Weyl dimension formula:*

$$\dim V_\lambda = \prod_{\alpha>0} \frac{\langle \alpha | \lambda + \rho \rangle}{\langle \alpha | \rho \rangle}.$$

In both cases, ρ denotes the half-sum of the positive roots α.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write an essay on the *Weyl integration formula* for the unitary group $\mathrm{U}(n)$, explaining its proof as well as its application to Weyl's formula for the irreducible characters. You should identify the key ingredients (maximal torus, Weyl group, roots, dominant weights).

2 Let G be a compact connected Lie group whose adjoint representation is irreducible. Show that, unless $G \simeq \mathrm{U}(1)$, the centre Z of G is finite and that any closed normal subgroup of G is contained in Z.

Let H be another connected Lie group and $\Phi : G \to H$ a Lie group homomorphism whose differential $d\Phi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is surjective on Lie algebras. Show that $H \simeq G/F$, for some finite subgroup $F \subset Z$.

Briefly define the Spin groups. By considering the Spin representation, or otherwise, find an isomorphism $\mathrm{SU}(4)/\{\pm\mathrm{Id}\} \simeq \mathrm{SO}(6)$.

[*General theorems from the course may be used without proof, if clearly stated. You may also assume without proof that the adjoint representation of* SO(6) *is irreducible.*]

3 Construct an isomorphism ϕ from the adjoint representation of SO(3) to the standard representation $\boldsymbol{R}^3$. Show that you can scale your isomorphism so that the Lie bracket is sent to the cross product of vectors: $\phi([\xi,\eta]) = \phi(\xi) \times \phi(\eta)$.

Show that the conjugation action of SU(2) on quaternions preserves the hyperplane of imaginary quaternions. Use this to construct a homomorphism from $\mathrm{SU}(2)/\{\pm\mathrm{Id}\}$ to SO(3) and explain why your map is in fact an isomorphism.

[*General results from the course may be used without proof, if clearly stated.*]

4 For $n \geq 3$, show that the tensor cube $(\boldsymbol{C}^n)^{\otimes 3}$ of the standard representation of $\mathrm{U}(n)$ decomposes into *four* irreducibles, as follows: the irreducible representations with highest weights $(3,0,\ldots,0)$ and $(1,1,1,0,\ldots,0)$ each appear once, while the irreducible representation with highest weight $(2,1,0,\ldots,0)$ appears twice. What happens when $n = 2$?

[*Hint: The Weyl dimension formula can be very useful here.*]

5 Write down (with explanation) the character of the (complexified) adjoint representation of SO($2n$). For $n > 2$, show that this is irreducible, and identify its highest weight. What happens for $n = 2$?

Verify that the character above agrees with that of $\Lambda^2(\boldsymbol{C}^{2n})$, the exterior square of the (complexified) standard representation $\boldsymbol{C}^{2n}$. Also explain this without recourse to computation.

Is the symmetric square $\mathrm{Sym}^2(\boldsymbol{C}^{2n})$ irreducible?

[*Any theorems from the course that you use must be clearly stated. You may find the dimension formula more helpful than the Weyl character formula.*]

6 For an integral weight μ of a given compact, connected Lie group, let $p(\mu)$ denote the number of distinct ways of expressing it as a sum of positive roots. (For example, for SU(2), $p(\mu) = 1$ when μ is even and non-negative, and 0 otherwise.)

1. Show that expanding the product leads to an identity of Fourier series

$$\sum_{\nu} p(\nu) e^{-\nu} \cdot \prod_{\alpha > 0} (1 - e^{\alpha}) = 1,$$

   where the sum ranges over all integral weights ν. (You need not study the convergence of the series and may manipulate them as formal sums.)

2. Using this result and the Weyl character formula, or by any other method, prove the *Kostant multiplicity formula*: the coefficient $m_\lambda(\mu)$ of e^{μ} in the character of the irreducible representation of highest weight λ is

$$\sum_{w \in W} \varepsilon(w) p\left(w(\lambda + \rho) - (\mu + \rho)\right);$$

   here, ρ is the half-sum of the positive roots.

[*You may specialise to the unitary group, but you will find it easier to argue in full generality.*]

END OF PAPER