UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 5 June, 2006 9 to 12

PAPER 16

ALGEBRAIC TOPOLOGY

*Attempt **FIVE** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(a) Show that the universal cover of any closed 3-manifold with finite fundamental group has the same integral homology groups as the 3-sphere.

(b) Show that the closed oriented surface of any genus $g \geqslant 1$ has a connected covering space of degree n, for any $n \geqslant 1$. Compute the genus of any such covering space.

2

(a) Let $n \geqslant 2$. Let $f : \mathbf{RP}^n \to \mathbf{RP}^n$ be a continuous map which is not surjective. Show that f induces the zero map on homology groups in positive degrees. [You may assume the usual formal properties of the degree in $\mathbf{Z}/2$ of a map between arbitrary closed manifolds, analogous to the degree in $\mathbf{Z}$ of a map between oriented closed manifolds.] Deduce that f lifts to a map $\mathbf{RP}^n \to S^n$. Finally, show that f is homotopic to a constant map.

(b) Show that for any map $f : X \to Y$ of nonzero degree between closed oriented n-manifolds, f induces surjections on rational homology.

3

(a) Let X be any odd-dimensional closed manifold. Show that X has Euler characteristic zero.

(b) Let V be a finite-dimensional vector space V with a nondegenerate alternating ($\langle x, x\rangle = 0$ for all $x \in V$) bilinear form. Show that V has even dimension.

(c) Let X be a closed orientable manifold of dimension congruent to 2 modulo 4. Using (b), show that X has even Euler characteristic. Give an example to show that this can fail if we omit the assumption that X is orientable.

4 Let p be a prime number. Define a "Bockstein homomorphism" $\beta : H^i(X, \mathbf{Z}/p) \to H^{i+1}(X, \mathbf{Z}/p)$, meaning a homomorphism which fits into a long exact sequence

$$\cdots \to H^i(X, \mathbf{Z}/p) \to H^i(X, \mathbf{Z}/(p^2)) \to H^i(X, \mathbf{Z}/p) \xrightarrow{\beta} H^{i+1}(X, \mathbf{Z}/p) \to \cdots.$$

Here $H^i(X, \mathbf{Z}/(p^2)) \to H^i(X, \mathbf{Z}/p)$ should be the obvious map associated to the group homomorphism $\mathbf{Z}/(p^2) \to \mathbf{Z}/p$ which takes 1 to 1.

Show that the Bockstein is always zero on $H^0(X, \mathbf{Z}/p)$, but that for every prime number p and every $i \geqslant 1$ there is a space X such that β is nonzero on $H^i(X, \mathbf{Z}/p)$. For any space X, show that

$$\beta(xy) = (\beta x)y + (-1)^i x(\beta y)$$

for all $x \in H^i(X, \mathbf{Z}/p)$ and $y \in H^j(X, \mathbf{Z}/p)$.

5 Let X and Y be topological spaces, R a commutative ring. Let $f : X \to Y$ be a covering space of degree $n < \infty$. Define a pullback map on homology, $f^* : H_i(Y, R) \to H_i(X, R)$. Your map should have the property that $f_* f^*(u) = nu$ for all $u \in H_*(Y, R)$. Deduce that for any covering space $f : X \to Y$ of finite degree, $f_* : H_*(X, \mathbf{Q}) \to H_*(Y, \mathbf{Q})$ is surjective.

Thus the Betti numbers increase when we pass from a space to a finite covering space. Give examples of double covering spaces $X \to Y$ to show that the Betti numbers of X can either be equal to those of Y, or strictly larger (in some dimension) than those of Y.

6

(a) Let $C_1, \ldots, C_n$ be simple closed curves on a closed orientable surface X of genus g. Suppose that any two of the curves are disjoint, and that the curves are linearly independent in the rational homology of X. Show that $n \leqslant g$.

(b) Define the *degree* of a smooth complex curve C (a closed complex submanifold of complex dimension 1) in $\mathbf{CP}^2$ to be the integer d such that C is homologous to d times a line $\mathbf{CP}^1$ in $\mathbf{CP}^2$. Show that the complex curve $\{[x, y, z] \in \mathbf{CP}^2 : x^d + y^d = z^d\}$ has degree d.

[*Hint: intersect it with a line.*]

END OF PAPER