UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 1 June, 2006 9 to 12

PAPER 15

DIFFERENTIAL GEOMETRY

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let (M, g) be an n-dimensional smooth Riemannian manifold with $n \geq 2$. Let $N \subset M$ be a 2-dimensional submanifold given by the image of a smooth map

$$\phi : (-\delta, \delta) \times (-\epsilon, \epsilon) \to M,$$

such that

$$t \mapsto \phi(t, s)$$

is a geodesic for all $s \in (-\epsilon, \epsilon)$. Denoting this curve as $\gamma_s : (-\delta, \delta) \to M$, derive the equation for the second variation of the length of γ_s, i.e. derive an equation for $\frac{d^2}{ds^2} L(\gamma_s)$ involving curvature. Deduce that the Gauss curvature of N at any $p \in N$ (N is given the induced metric from M) is less than or equal to the sectional curvature of M at p evaluated for the 2-plane in T_pM that is tangent to N. Show by explicit example that equality need not hold.

2 Let (M, g), (N, h) denote connected smooth Riemannian manifolds. Define the notion of *isometry* and *local isometry*. Suppose ϕ_t is a one-parameter family of local transformations of (M, g) which are local isometries. Let K denote the vector field generating ϕ_t. Show that K satisfies the *Killing equation*:

$$g(\nabla_X K, Y) + g(\nabla_Y K, X) = 0$$

for all vector fields X, Y. Vector fields satisfying the Killing equation for all vector fields X, Y are known as *Killing fields.* Conversely, show that a Killing field K generates a 1-parameter family of local isometries.

For vector fields X, Y and Z, define $\nabla^2_{X,Y} Z = \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z$. Show that for K a Killing field, we have

$$g(\nabla^2_{X,Y} K, Z) + g(\nabla^2_{X,Z} K, Y) = 0.$$

Now show that

$$\nabla^2_{X,Y} K = R(K, X)Y,$$

where R denotes the Riemann tensor. [*Hint: What is* $\nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z$*?*] Finally, deduce from the above or otherwise that if K and $\tilde{K}$ are Killing and $K(p) = \tilde{K}(p)$, $\nabla_z K(p) = \nabla_z \tilde{K}(p)$ for some point $p \in M$, and all $z \in T_pM$, then $K = \tilde{K}$ identically.

3 Let (M, g) be a smooth connected Riemannian manifold. A *geodesically convex neighorhood* U of a point p is a neighborhood for which there exists an $\epsilon > 0$ such that all points in U can be joined by a unique length minimizing geodesic of length less than or equal to ϵ, contained completely in U. Recall that around each point p, there exists a geodesically convex neighborhood U.

Define the induced metric space structure of M and show that it indeed defines a metric space. Define the notion of *geodesic completeness*. Prove the Hopf-Rinow theorem: M is geodesically complete if and only if M is complete as a metric space. Prove that the real hyperbolic 2-plane $\mathbb{H}^2$ is geodesically complete.

4 Let (M, g) be an n-dimensional smooth connected Riemannian manifold with $n \geq 2$, and let $\gamma : [0, L] \to M$ be a geodesic parametrized by arc length. Let $\mathbf{V}$ denote the set of piecewise smooth vector fields along γ which vanish at $\gamma(0)$ and $\gamma(L)$, and which are perpendicular to γ.

State the definition of the *index form* $I : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$. State the definition of *Jacobi field* and *conjugate point*. Give without proof a characterization of when I is positive definite and strictly positive definite in terms of conjugate points.

Now, let (M, g) satisfy $Ric(v, v) \geq (n-1)\kappa g(v, v)$ for some constant $\kappa > 0$ and all vectors v. If $L > \pi/\sqrt{\kappa}$, show that there exists a $V \in \mathbf{V}$ with $I(V, V) < 0$. Deduce that if in addition M is assumed to be complete, then $\text{diam}(M) \leq \pi/\sqrt{\kappa}$. Show by explicit example that the above inequality may be violated if M is not complete.

END OF PAPER