UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 7 June, 2006 9 to 11

PAPER 14

PROBABILISTIC COMBINATORICS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Suppose that $G = G(n,p)$ is a random graph and let μ be its expected chromatic number. Prove that

$$\mathbb{P}(|\chi(G) - \mu| > \lambda\sqrt{n}) < 2e^{-\lambda^2/2}.$$

Now suppose that p is such that

$$\mathbb{P}(\chi(G) < n/\log n) > 1/10.$$

Prove that $\chi(G) < n/\log n + \log n\sqrt{n}$ with high probability.

[You may assume basic results about conditional expectation, and that the vertex exposure martingale corresponding to the chromatic number is Lipschitz with constant 1. However, you should prove the Azuma-Hoeffding Inequality if you use it.]

2 Show that there exists a 3-uniform hypergraph H with the following properties:

1. H has chromatic number at least 2006.
2. H does not contain two edges sharing two vertices.

[You may assume any standard results if clearly stated.]

3 Prove that every r-uniform hypergraph with less than 2^{r-1} edges can be two-coloured. Further prove that there exists an r-uniform hypergraph with $O(r^2 2^r)$ edges that is not two-colourable.

4 Suppose that $G = G(n,p)$ where $p = (\log n + c)/n$ and c is a constant. State with proof the limiting probability that

a. there exists a vertex of degree zero;

b. there are exactly two vertices of degree zero;

c. the graph is connected.

Let X be the number of vertices of degree one in G. By showing that $\mathrm{Var}(X)/(\mathbb{E}X)^2 \to 0$ or otherwise show that there is a vertex of degree one with high probability.

[Any standard results about convergence in distribution may be assumed if they are clearly stated. You may also assume that, with high probability, there is no component of size between $\log\log n$ and $n/2$.]

5 State the general form of the Lovász Local Lemma and deduce the symmetric form from it.

Suppose that H is an r-uniform hypergraph on a (finite) vertex set V and that the maximum degree of H is Δ. Prove that H is two-colourable provided that $\Delta < 2^r/(2re)-1$.

Now consider a random two colouring of the vertices of H where each vertex is coloured red or blue independently with probability one half.

i) Fix an edge W of H. Prove that the probability that W contains more than $\frac{3}{4}r$ red points is less than $e^{-r/8}$.

ii) Prove that, provided $\Delta < e^{r/8}/(2re) - 1$, there is a two-colouring of V such that no edge of H contains more than $\frac{3}{4}r$ points of either colour.

[You may assume any tail estimates if clearly stated.]

6 Suppose that A and B are up-sets in $\{0,1\}^n$ and that $Z_1, Z_2, \ldots Z_n$ are independent Bernoulli random variables with $\mathbb{P}(Z_i = 1) = p_i$. Let $Z = (Z_1, Z_2, \ldots Z_n)$. Show that

$$\mathbb{P}(Z \in A \cap B) \geq \mathbb{P}(Z \in A)\mathbb{P}(Z \in B)$$

and

$$\mathbb{P}(Z \in A \Box B) \leq \mathbb{P}(Z \in A)\mathbb{P}(Z \in B).$$

Suppose that $G = G(n,p)$, that U, W are disjoint subsets of vertices and that x is a vertex not in U or W. Further suppose that the probability that G contains a path from x to U is α and from x to W is β. Prove that the probability that there is a path from U to W through x is at most $\alpha\beta$ but that the probability that there is a path from U to W is at least $\alpha\beta$.

END OF PAPER