UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 1 June, 2006 9 to 11

PAPER 13

QUASIRANDOMNESS

*Attempt **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let X and Y be sets of size m and n, respectively. Let $f : X \times Y \to [-1, 1]$.

Prove that the following two properties of f are equivalent, in a sense that you should explain:

(a)

$$\sum_{x,x' \in X} \sum_{y,y' \in Y} f(x,y)f(x,y')f(x',y)f(x',y') \leqslant c_1 m^2 n^2.$$

(b) For every subset $A \subset X$ and every subset $B \subset Y$,

$$\Big|\sum_{x \in A} \sum_{y \in B} f(x,y)\Big| \leqslant c_2 mn.$$

Prove that there exists a function f taking values ± 1 such that properties (a) and (b) hold, with constants c_i that converge to 0 as m and n tend to infinity.

2 State and prove some version of Szemerédi's regularity lemma. Use it to prove that for every $\delta > 0$ there exists N such that for every subset $A \subset \{1, 2, \ldots, N\}^2$ of size at least δN^2 there exist x, y and $d \neq 0$ such that (x, y), $(x, y + d)$ and $(x + d, y)$ belong to A.

3 (i) Let A, B, C be subsets of $\mathbb{Z}_N$ with $|A| = \alpha N$, $|B| = \beta N$ and $|C| = \gamma N$. Suppose that the number of quadruples $(x, y, z, w) \in A^4$ such that $x + y = z + w (\text{mod } N)$ is at most $(\alpha^4 + c)N^3$. Prove that the number of triples $(a, b, c) \in A \times B \times C$ such that $a + c = 2b (\text{mod } N)$ differs from $\alpha\beta\gamma N^2$ by at most $f(c)$, where $f(c) \to 0$ as $c \to 0$.

(ii) Let $\delta > 0$ and let $A \subset \{1, 2, \ldots, N\}$ be a set of size at least δN. Prove that $A + A + A$ contains an arithmetic progression of length at least N^r, where $r > 0$ depends on δ only.

END OF PAPER