UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Monday 5 June, 2006 1.30 to 3.30

## PAPER 12

## COMBINATORICS

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

(i) State and prove the Local LYM inequality, and deduce the LYM inequality. Which antichains in $\mathcal{P}([n])$ have size exactly $\binom{n}{\lfloor n/2 \rfloor}$?

(ii) A *symmetric chain* in $\mathcal{P}([n])$ is a chain $A_1 \subset \ldots \subset A_k$ $(k \geqslant 1)$ such that $|A_{i+1}| = |A_i| + 1$ for each $1 \leqslant i \leqslant k-1$ and also $|A_1| + |A_k| = n$. Prove that $\mathcal{P}([n])$ may be partitioned into symmetric chains. [*Hint: induction on $n$.*]

**2** State the vertex-isoperimetric inequality in the discrete cube (Harper's theorem). Explain carefully how the Kruskal-Katona theorem may be deduced from Harper's theorem.

State the Erdős-Ko-Rado theorem, and give two proofs: one using the Kruskal-Katona theorem and one using cyclic orderings.

Which of the following are always true, for every $n$ and every $r \leqslant n/2$, and which can be false? Give proofs or counterexamples as appropriate.

(i) If $\mathcal{A} \subset [n]^{(r)}$ is an intersecting family then the initial segment of the lexicographic order on $[n]^{(r)}$ of size $|\mathcal{A}|$ is also intersecting.

(ii) If $\mathcal{A} \subset [n]^{(r)}$ is an intersecting family then the initial segment of the colexicographic order on $[n]^{(r)}$ of size $|\mathcal{A}|$ is also intersecting.

(iii) If $\mathcal{A} \subset [n]^{(r)}$ is a 2-intersecting family then the initial segment of the lexicographic order on $[n]^{(r)}$ of size $|\mathcal{A}|$ is also 2-intersecting.

**3** State and prove the edge-isoperimetric inequality in the discrete cube (the theorem of Harper, Lindsey, Bernstein and Hart).

Deduce that the isoperimetric number of the discrete cube is 1.

Which subsets of size $2^{n-1}$ of the discrete cube $Q_n$ have edge-boundary of size exactly $2^{n-1}$? Justify your answer.

**4** State and prove the Frankl-Wilson theorem (on modular intersections).

Let $p$ be prime. Using the Frankl-Wilson theorem, show that if $A \subset [4p]^{(2p)}$ satisfies $|x \cap y| \neq p$ for all $x, y \in A$ then $|A| \leqslant 2\binom{4p}{p-1}$.

Explain the Kahn-Kalai counterexample to Borsuk's conjecture.

Give, with justification, an explicit $n$ such that Borsuk's conjecture is false in dimension $n$.

## END OF PAPER