UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 7 June, 2006 1.30 to 4.30

PAPER 10

ADVANCED COMPLEX VARIABLE

*Attempt **four** questions.*

*There are **six** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define a normal family. Show that the family of analytic functions $f : \Omega \to \mathbb{D}$ from a plane domain Ω into the unit disc $\mathbb{D}$ is a normal family.

Let $g : H = \{x + iy : y > 0\} \to \mathbb{D}$ be an analytic function. Suppose that

$$g(iy) \to \ell \qquad \text{as} \qquad y \searrow 0$$

for some finite limit ℓ. Show that the family of maps $z \mapsto g(z/n)$ form a normal family and deduce that $g(z) \to \ell$ as $z \to 0$ non-tangentially.

Give an example to show that we need not have $g(z) \to \ell$ as $z \to 0$, where g is as in the previous paragraph.

2 Let Ω be a plane domain and (z_n) a discrete set of points in Ω. Prove that there is an analytic function $f : \Omega \to \mathbb{C}$ which has zeros at the points (z_n) and no other zeros.

Let q be a complex number with $|q| > 1$. For which values of $z \in \mathbb{C}$ does the infinite product

$$\sigma(z) = \prod_{k=0}^{\infty} (1 - q^{-2k-1}z)(1 - q^{-2k-1}z^{-1})$$

converge? Where are the zeros of σ? Prove that

$$\sigma(qz) = -z\sigma(q^{-1}z)$$

wherever both sides are defined.

3 Explain briefly how $\mathbb{P} \setminus \{0, 1, \infty\}$ can be regarded as a quotient of the upper half-plane by a group of conformal maps that acts discontinuously. Explain how this leads to a hyperbolic metric on $\mathbb{P} \setminus \{0, 1, \infty\}$.

Prove the Schwarz – Pick lemma and deduce that any analytic function $f : \mathbb{D}\setminus\{0\} \to \mathbb{P}\setminus\{0, 1, \infty\}$ is a contraction for the hyperbolic metric. Is it ever the case that the hyperbolic distance between two distinct points $z_0, z_1 \in \mathbb{D} \setminus \{0\}$ is equal to the hyperbolic distance between $f(z_0)$ and $f(z_1)$?

4 Let K be a compact subset of the plane domain Ω and let $\mathcal{O}(\Omega)$ be the vector space of all analytic functions $f : \Omega \to \mathbb{C}$. Define the *holomorphic hull* $\widehat{K}$ of K in Ω.

State and prove Runge's Theorem.

Prove that, for any point $w \in \Omega$, we have

$$w \in \widehat{K}$$

if, and only if,

$$|f(w)| \leqslant \sup\{|f(z)| : z \in K\} \qquad \text{for all } f \in \mathcal{O}(\Omega) \ .$$

5 Let $\phi : \mathbb{T} \to \mathbb{C}$ be a continuous function on the unit circle and let m denote the Lebesgue measure on $\mathbb{T}$ normalised so $m(\mathbb{T}) = 1$. Show that the Poisson integral

$$P\phi : z \to \int_{\mathbb{T}} \mathrm{Re} \left(\frac{\omega + z}{\omega - z} \right) \phi(\omega) \, dm(\omega)$$

defines a harmonic function on the unit disc with $P\phi(z) \to \phi(\omega)$ as $z \to \omega \in \mathbb{T}$.

The *Cauchy transform* of ϕ is the function

$$\Phi : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C} \ ; \quad w \mapsto \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\phi(z)}{z - w} \, dz \ .$$

Show that this defines an analytic function on $\mathbb{C} \setminus \mathbb{T}$. Deduce that the jump in Φ across the unit circle is given by ϕ in the sense that

$$\Phi(r\omega) - \Phi(r^{-1}\omega) \to \phi(\omega) \qquad \text{as} \qquad r \searrow 1$$

for each ω with $|\omega| = 1$.

6 Define the Hardy–Littlewood maximal function and prove the maximal inequality.

Let f be a Lebesgue integrable function on the unit circle. Prove that, for Lebesgue almost every point $\omega \in \mathbb{T}$, we have

$$\frac{1}{m(I)} \int_I f \, dm \to f(\omega)$$

as the interval I, centred on ω, shrinks to $\{\omega\}$.

Show that the function f need not have such a limit at every point of the unit circle.

END OF PAPER