UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 1 June, 2006 1.30 to 4.30

# PAPER 1

# LIE ALGEBRAS AND REPRESENTATION THEORY

*Attempt **QUESTION 1** and **THREE** other questions.*

*There are **SIX** questions in total.*

*The questions carry equal weight.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*5 sheets of triangular graph paper*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1**

(a) Let $V = \langle x, y \rangle$ be the standard representation of $\mathfrak{sl}_2$. For the representation

$$U = S^3 V \otimes S^2 V$$

- give a basis for each of the weight spaces of $U$;
- draw the weight diagram of $U$;
- identify all highest weight vectors;
- for the submodule isomorphic to $\Gamma_1$, give a basis explicitly.

(b) Let $W = \langle e_1, e_2, e_3 \rangle$ be the defining representation of $\mathfrak{sl}_3$. Let

$$Z = W \otimes W \otimes W^*$$

- give a basis for each of the weight spaces of $Z$;
- draw the weight diagram of $Z$;
- draw the weight diagrams for all of the irreducible submodules of $Z$;
- identify the highest weight vector of the submodule with highest weight $-2L_3$.
- give an explicit basis for the submodule with highest weight $2L_1 - L_3$.

**2** State and prove Cartan's criterion. Results about nilpotent Lie algebras, and the Jordan decomposition of matrices should be clearly stated.

In two sentences or fewer explain the significance of these results for the classification theorem for semi-simple Lie algebras.

**3**

(a) Let $G \subset \mathbb{R}^N$ be a Lie group. Define the Lie algebra of $G$ (as a set). Show that the Lie algebra of $SL_n$ is contained in $\mathfrak{sl}_n$.

(b) Explain how the Lie bracket is defined on the Lie algebra of $G$. Using this definition, choose curves representing $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in $\mathfrak{sl}_2$, and show explictly that the bracket defined in this way satisfies $[X, Y] = -[Y, X]$.

(c) Explain what is meant by a vector field on a smooth manifold $M$. Explain what is meant by a left invariant vector field on a Lie group $G$.

Show that an element $X$ in the Lie algebra of $G$ gives rise to a left invariant field $v_X$ on $G$. For $G = \mathfrak{sl}_2$, calculate $v_X\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)$ explicitly when $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

**4**

(a) Let $\mathfrak{g}$ be a Lie algebra. Define the Killing form $B(\ ,\ )$ on $\mathfrak{g}$ and show that it is an invariant symmetric form.

(b) Now assume $\mathfrak{g}$ is semisimple, with Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha$$

Show that $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) \neq 0 \Leftrightarrow \alpha = -\beta$. Show that $B$ restricted to $\mathfrak{h}$ is non-degenerate. You may assume that $B(\ ,\ )$ is non-degenerate.

(c) For $\mathfrak{g} = \mathfrak{sl}_3$ compute

$$B(H_{12}, H_{23})$$

(You may state any results about the Cartan decomposition you wish to use.)

(d) Let $V$ be an irreducible representation of $\mathfrak{sl}_2$ with highest weight $k$. With $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as usual, show that for $v \in V$

$$XY^n v = n(k - n + 1)Y^{n-1} v$$

and thus write down an expression for $tr(XY)$ on $V$.

(e) For $\alpha$ a root, define $\mathfrak{s}_\alpha$.

On $\mathfrak{sl}_3$ paper, draw the weight diagram for the adjoint representation of $\mathfrak{sl}_3$.

Now restrict the adjoint representation to the sub algebra $\mathfrak{sl}_{3_{L_1 - L_2}}$.

Indicate which weights of $\mathfrak{sl}_3$ belong to which irreducible representations of $\mathfrak{sl}_{3_{L_1 - L_2}}$.

Hence calculate $B(E_{12}, E_{21})$.

**5**

(a) Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_\alpha$ be the Cartan decomposition of a semisimple Lie algebra $\mathfrak{g}$.

Define the weight lattice $\Lambda_W$.

Define the Weyl group $\mathfrak{W}$ and show that for $W \in \mathfrak{W}$

$$W : \Lambda_W \to \Lambda_W .$$

(b) Define the fundamental Weyl chamber $\mathcal{W}$ and show that for $\beta$ in $\Lambda_W$ there exists $W$ in $\mathfrak{W}$ such that $W\beta \in \overline{\mathcal{W}}$.

(c) Compute $W$ in the Weyl group of $\mathfrak{s}l_3$ such that

$$W(3L_2 - 2L_1) \in \mathcal{W}.$$

What is $W(3L_2 - 2L_1)$?

(Use the standard set of positive roots. The computation may be illustrated on $\mathfrak{s}l_3$ paper or calculated algebraically as you wish.)

**6** Let $\mathbb{E}$ be a real vector space with inner product ( , )

(a) Give the definition of an abstract root system $R \subset \mathbb{E}$.

(b) For $\alpha,\ \beta \,\varepsilon\, R$, with $(\alpha, \beta) < 0$ list all the possibilities for

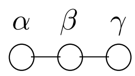
$$n_{\alpha,\beta} = 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}$$

For each $n_{\alpha,\beta}$, give the value of $n_{\beta,\alpha}$.

(c) Show that if $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root.

[*Hint: if $W_\alpha$ ($W_\beta$) is reflection across $\alpha^\perp$ ($\beta^\perp$) consider $W_\alpha\beta$ and $W_\beta\alpha$.*]

(d) Explain how an ordering is put on $R$. Define $R_+$ and show that $R = R_+ \sqcup R_-$. Explain what is meant by simple roots and show that $\alpha, \beta$ simple $\Rightarrow \alpha - \beta$ not a root.

(e) List all positive roots of the root system whose Dynkin diagram is 


Show that this list is complete.

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