UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 13 June, 2005 1.30 to 4.30

PAPER 86

VALUE DISTRIBUTION OF ANALYTIC FUNCTIONS

Attempt ***THREE*** *questions.*

There are ***SIX*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State and prove the argument principle for holomorphic maps.

Let f and (f_n) for $n \in \mathbb{N}$ be holomorphic maps on a domain $\Omega \subset \mathbb{C}$ with f_n converging to f locally uniformly on Ω. Assume that f is not constant. If $f(z_o) = 0$ for some point $z_o \in \Omega$, show that there are points $z_n \in \Omega$, for n sufficiently large, with

$$z_n \to z_o \qquad \text{as} \;\; n \to \infty \qquad \text{and} \;\; f_n(z_n) = 0 \; .$$

If $f(z_o) = f'(z_o) = 0$, can we always choose the points z_n so that $f_n(z_n) = f_n'(z_n) = 0$ for sufficiently large n?

2 Define the hyperbolic metric on $\mathbb{D}$ and explain when a Riemann surface has a hyperbolic metric. Prove that a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ is a contraction for the hyperbolic metric. State and prove a similar result for analytic maps $g : R \to S$ between two Riemann surfaces where S has a hyperbolic metric.

Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map with $f(z_o) = w_o$. Show that f may be written as

$$f(z) = \frac{(z - z_o)h(z) + w_o(1 - \overline{z_o}z)}{(z - z_o)\overline{w_o}h(z) + (1 - \overline{z_o}z)}$$

for some holomorphic function $h : \mathbb{D} \to \mathbb{C}$ with $\sup\{|h(z)| : z \in \mathbb{D}\} \leqslant 1$.

Let $\mathcal{F}$ be the set of all holomorphic functions $f : \mathbb{D} \to \mathbb{D}$ with $f(z_o) = w_o$. Show that

$$\Delta(z) = \{f(z) : f \in \mathcal{F}\}$$

is a closed disc in $\mathbb{D}$ for each $z \in \mathbb{D}$.

3 Define the deficiency $\delta(a)$ and ramification index $\theta(a)$ for a point $a \in \mathbb{P}$.

Show that the exponential function $e : z \mapsto e^z$ has Nevanlinna characteristic $T_e(R)$ with

$$T_e(R) \sim cR$$

for some constant $c > 0$. If $U : \mathbb{P} \to \mathbb{P}$ is a rational function of degree d, find a similar formula for the Nevanlinna characteristic of $f = U \circ e$.

Using the above result, or otherwise, construct a meromorphic function $f : \mathbb{C} \to \mathbb{P}$ and a point $a \in \mathbb{P}$ for which both $\delta(a)$ and $\theta(a)$ are non-zero.

4 Explain what is meant by the *order* $\mathrm{ord}\, f$ of a meromorphic function $f : \mathbb{C} \to \mathbb{P}$ and the *order* of the counting function $N(R; a)$.

Suppose that the meromorphic function f has finite order. Show how to deduce from Nevanlinna's Theorems that the order of $N(R; a)$ must be equal to the order of f except for at most two exceptional values of a. Give an example to show that there can be exceptional values.

Let $U : \mathbb{P} \to \mathbb{P}$ be a Möbius transformation. Prove that the order of $U \circ f$ is the same as that of f. Prove that the sum $f = f_1 + f_2$ of two meromorphic functions satisfies

$$\mathrm{ord}\, f \leqslant \max\{\mathrm{ord}\, f_1, \mathrm{ord}\, f_2\} \ .$$

Show that, when $\mathrm{ord}\, f_1 \neq \mathrm{ord}\, f_2$ then $\mathrm{ord}\, f = \max\{\mathrm{ord}\, f_1, \mathrm{ord}\, f_2\}$ but, when $\mathrm{ord}\, f_1 = \mathrm{ord}\, f_2$, the inequality may be strict.

5 Prove that a continuously differentiable, increasing function $\phi : [0, \infty) \to [0, \infty)$ satisfies

$$\phi'(x) \leqslant \phi(x)^2$$

except on a set $I \subset [0, \infty)$ with finite measure. Explain briefly how this result is used in the proof of Nevanlinna's Second Theorem.

Let $f : \mathbb{C} \to \mathbb{P}$ be a meromorphic function that is not rational. Show that, for any five distinct points a_1, a_2, a_3, a_4, a_5 in $\mathbb{C}$ there must be at least one with $f(z) - a_k$ having a simple zero (that is, a zero of degree 1). Need there be more than 1 simple zero? Need there be more than 1 point a_k with $f(z) - a_k$ having a simple zero?
(You may assume any result from the course, provided that it is stated clearly.)

6 Let $g : \mathbb{D} \to \mathbb{D}$ be a holomorphic map from the unit disc to itself. Write

$$\|g'(z)\| = \frac{2|g'(z)|}{1-|g(z)|^2} \ .$$

The *hyperbolic characteristic* $T_{\mathbb{D}}(R)$ is given by

$$T_{\mathbb{D}}(R) = \int_{D(0,R)} \left(\frac{1}{2\pi} \log \frac{R}{|z|} \right) \|g'(z)\|^2 \, d\mathbb{A}_{\mathbb{C}}(z)$$

and the counting function for critical points is

$$N_1(R) = \sum \left\{ (\deg g(z) - 1) \log \frac{R}{|z|} : |z| < R \text{ and } g'(z) = 1 \right\} \ .$$

Prove that

$$T_{\mathbb{D}}(R) + N_1(R) = \int_0^{2\pi} \log \|g'(Re^{i\theta})\| \, \frac{d\theta}{2\pi} - \log \|g'(0)\| \ .$$

Deduce that

$$T_{\mathbb{D}}(R) + N_1(R) \leqslant C_1 \rho(R) + C_2$$

where $\rho(R)$ is the hyperbolic distance in $\mathbb{D}$ from 0 to R and C_1, C_2 are constants that may depend on g but not on R.

END OF PAPER