UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 3 June, 2005 1.30 to 4.30

PAPER 85

NOETHERIAN ALGEBRAS

*Attempt **TWO** questions from each section.*

*There are **THREE** questions in each section.*

All questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Let R be a ring. Show that the intersection J of all maximal left ideals of R is equal to the intersection of all maximal right ideals.

Show that J contains every nilpotent ideal of R.

Compute J in the case when R is the power series ring $\mathbb{Z}_p[[t]]$, where $\mathbb{Z}_p$ is the ring of $p-$adic integers. Briefly justify your answer.

2 Let R be a right Noetherian ring. Show that $R[t]$ is also right Noetherian.

Now let k be a field and let R be an almost commutative $k-$algebra.

(a) Show that there exists a finite dimensional $k-$Lie algebra $\mathfrak{g}$ such that R is a quotient of $\mathcal{U}(\mathfrak{g})$.

(b) Show that R is right and left Noetherian.

3 Let R be a ring, S a multiplicatively closed subset of R containing 1. What is a right localisation of R at S? State necessary and sufficient conditions for the right localisation of R at S to exist, and prove their necessity.

Prove that if R is a right Noetherian domain, then a right localisation of R at $S = R\backslash\{0\}$ exists.

State Goldie's Theorem and use it to deduce that any prime right Noetherian ring has a right classical ring of quotients of the form $M_n(D)$ for some division ring D.

Give an example of a ring which has a right classical ring of quotients, but which is not right Noetherian.

SECTION B

4 Let R be a right Artinian ring and let $J = J(R)$.

(a) Show that if $J = 0$ then R_R is semisimple.

(b) Show that J is nilpotent.

(c) Explain briefly how these results can be used to show that R is right Noetherian.

Give an example of a right Artinian ring which is not left Noetherian. Justify your answer.

5 Let R be a commutative ring. Show that R is semiprime if and only if R has no nonzero nilpotent elements.

Suppose further that R is Noetherian. For any prime ideal P of R, show that the localisation R_P is a Noetherian local ring.

Suppose R is such that R_Q is semiprime for all prime ideals Q of R. Show that R is also semiprime. Is this statement true when "semiprime" is replaced with "prime"? Give a proof or a counterexample.

6 Let k be a field and let $R = A_n(k)$ be the $n-$th Weyl algebra, $n \geqslant 1$. Show that the following are equivalent:

(a) R is a simple ring,

(b) R has no nonzero modules which are finite dimensional over k,

(c) $\text{char}(k) = 0$.

State and prove Bernstein's inequality concerning modules for $A_n(k)$.

END OF PAPER