UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 8 June, 2005 9 to 11

PAPER 83

SUPERFLUID VORTICES

Attempt ***TWO*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Dynamics of a Bose-Einstein condensate is described by the Gross-Pitaevskii equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V_{ext}(\mathbf{x})\psi + U|\psi|^2\psi, \quad (1)$$

where $\hbar$ is the Plank's constant, m is the particle mass, V_{ext} is an external potential (if any), and U is the effective pair interaction. The number density is $n(\mathbf{x}) = |\psi(\mathbf{x})|^2$. The total number of particles is $N = \int |\psi(\mathbf{x})|^2\, d\mathbf{x}$.

(i) Write down the Hamiltonian functional $H[\psi, \psi^*]$ for the Gross-Pitaevskii equation and derive the equation for the ground state of the condensate.

(ii) Show that a dimensionless form of Eq. (1) without the external potential, $V_{ext}(\mathbf{x}) = 0$, can be written as

$$-2i\psi_t = \nabla^2\psi + (1 - |\psi|^2)\psi. \quad (2)$$

What are the units of length and time? What is the ground state?

(iii) Write down the equation for the solitary waves moving with velocity v in the positive z-direction in a condensate described by Eq. (2) in the frame of reference in which the solitary wave is stationary. Write down the linearised equations for the disturbances of the real and imaginary parts of ψ with respect to the ground state. Assume that the linearised equations are satisfied by sinusoidal disturbances of wavenumber k and find v as a function of k.

(iv) The wave function of a vortex ring that moves with velocity u along the straight-line vortex (with winding number $\mathcal{N} = 1$) positioned along the $z-$axis can be written as

$$\psi = [R(s) + \phi(s, z)] \exp[i\theta],$$

where (s, θ, z) are cylindrical coordinates and $R(s)$ is the amplitude of the straight-line vortex. Show that the equation on $\phi(s, z)$ in the frame of reference moving with the vortex ring can be written as

$$2\mathrm{i}u\frac{\partial\phi}{\partial z} = \frac{1}{s}\frac{\partial}{\partial s}\left[s\frac{\partial\phi}{\partial s}\right] + \frac{\partial^2\phi}{\partial z^2} - \frac{\phi}{s^2}$$
$$+ (1 - 2R^2 - R(\phi + 2\phi^*) - |\phi|^2)\phi - R^2\phi^*.$$

2 The wave function of a Bose-Einstein condensate in an optical trap is described by

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\nabla^2\psi + V_{ext}(\mathbf{x})\psi + U|\psi|^2\psi, \tag{3}$$

with $V_{ext}(\mathbf{x}) = sE_R \sin^2(\pi x/d)$, where s is the lattice depth, d is the lattice period, E_R is the recoil energy and U is the effective interaction potential.

(i) If there are no interactions between particles ($U = 0$) prove that

$$\psi_{jk}(x + d) = \exp[ikd]\psi_{jk}(x),$$

where ψ_{jk} is the Bloch function of band j and quasi-momentum $\hbar k$.

(ii) Given the ground state $\psi_0(x)$ of the system Eq. (3) with $U \neq 0$, write down the expression for the chemical potential μ. Define the compressibility κ of the system Eq. (3). What is the sign of $\frac{\partial\kappa}{\partial s}$?

(iii) Consider a radially symmetric "lattice" $V_{ext}(\mathbf{x}) = sE_R \sin^2(\pi r/d)$, where $r^2 = x^2+y^2$. In the steady state the vortex line of winding number $\mathcal{N}$ is centred at the origin $r = 0$ and parallel to z-axis. Write down the equation on the amplitude $R(r)$ of the vortex wave function in polar coordinates (r, θ).

(iv) If $s = 0$, re-write the equation on the amplitude in dimensionless form using the healing length $l_0 = \hbar/\sqrt{2m\mu}$ as the unit length and scaling R, so that $\tilde{R}(\tilde{r}) \to 1$ as $\tilde{r} \to \infty$, where tilde stands for dimensionless units. Evaluate

$$\lim_{\tilde{r}\to\infty} \tilde{r}^2(1 - \tilde{R}(\tilde{r})).$$

(v) Plot the graph of $R(r)$ for $s \neq 0$.

3 A modification of the Gross-Pitaevskii equation that included the mutual friction with normal fluid moving with velocity $\mathbf{v}_n$ can be written in dimensionless form as

$$-i\psi_t = \tfrac{1}{2}\nabla^2\psi + (1-|\psi|^2)\psi - \zeta\psi\Big(\partial_t + \mathbf{v}_n\cdot\nabla\Big)|\psi|^2, \tag{4}$$

where ζ is a given positive dissipation coefficient.

(i) Show that Eq. (4) is Galilean invariant that is, consider a frame of reference with coordinate $\mathbf{x}'$, moving with a velocity $\mathbf{u}$ with respect to the original frame. Show that ψ should transform as

$$\psi'(\mathbf{x}',t) = \psi(\mathbf{x},t)\exp(\mathbf{A}\cdot\mathbf{x} - Bt),$$

and identify $\mathbf{A}$ and B.

(ii) Consider a family of slowly varying solutions to Eq. (4)

$$\psi = U(\mathbf{X},T,\epsilon)\exp\Big(i\epsilon^{-1}\Theta(\mathbf{X},T,\epsilon)\Big), \quad \mathbf{v}_n = \mathbf{v}_n(\mathbf{X},T,\epsilon), \quad \nabla\cdot\mathbf{v}_n = 0,$$

where $\mathbf{X} = \epsilon\mathbf{x}$, $T = \epsilon t$ and $0 < \epsilon \ll 1$. Show that to $O(\epsilon^2)$ the Gross-Pitaevskii equation with dissipation Eq. (4) reduces to the superfluid Euler equation with bulk viscosity ξ

$$\tfrac{\partial\mathbf{v}_s}{\partial t} + \tfrac{1}{2}\nabla|\mathbf{v}_s|^2 = -\nabla\mu + \xi\nabla\nabla\cdot\rho_s(\mathbf{v}_s - \mathbf{v}_n).$$

What are μ, ξ, ρ_s and $\mathbf{v}_s$ in terms of U, ζ and Θ?

(iii) Assume that the normal fluid is at rest, $\mathbf{v}_n = 0$. Find the plane wave solutions. Consider the perturbed plane wave $\psi = Ue^{i\theta}$ with

$$U(\mathbf{x},t) = 1 + \epsilon R(\mathbf{x},t), \quad \text{and} \quad \theta(\mathbf{x},t) = \mathbf{k}\cdot\mathbf{x} - \tfrac{1}{2}k^2t + \epsilon\Theta(\mathbf{x},t).$$

Assume also spatially periodic solutions of the form

$$(R,\Theta) = (R_0,\Theta_0)e^{i\mathbf{l}\cdot\mathbf{x}+\sigma t}.$$

Find the equation for the growth exponent σ.

4 Consider a mixture of two different species of bosons with masses m_1 and m_2. The condensate wave functions for the two components are ψ_1 and ψ_2 and the energy functional is given by

$$E = \int \Big\{ \tfrac{\hbar^2}{2m_1}|\nabla\psi_1|^2 + V_1(\mathbf{x})|\psi_1|^2 + \tfrac{\hbar^2}{2m_2}|\nabla\psi_2|^2 + V_2(\mathbf{x})|\psi_2|^2 + \tfrac{U_{11}}{2}|\psi_1|^4 + \tfrac{U_{22}}{2}|\psi_2|^4 + U_{12}|\psi_1|^2|\psi_2|^2 \Big\}\, d\mathbf{x},$$

where U_{ij} is the effective interaction potential for an atom of species i with one of species j and V_i are the external potentials acting on species i.

(i) Write down two stationary Gross-Pitaevskii equations for the equilibrium states of species 1 and 2 assuming that the interaction conserves separately the number of atoms of the two species.

(ii) For a homogeneous gas ($V_1 = V_2 = 0$) relate the chemical potentials μ_i to densities of the ground state $n_i = |\psi_i|^2$.

(iii) For the homogeneous solution to be stable, the energy must increase for deviations of the density from uniformity. Assume that the spatial scale of the density disturbances is so large that the kinetic energy term in the energy functional plays no role. Consider the change in total energy arising from small changes δn_1 and δn_2 in the ground state densities of the two components. From this, or otherwise, deduce the sufficient conditions for stability of the system in terms of U_{11}, U_{12} and U_{22}.

(iv) Now assume that the trapping potentials are isotropic and harmonic:

$$V_i(\mathbf{x}) = \tfrac{1}{2}m_i\omega_i^2 r^2, \quad i = 1, 2,$$

where ω_i are oscillators frequencies and $r^2 = |\mathbf{x}|^2$. Show that the density distributions in the Thomas-Fermi approximation are

$$n_1 = \frac{\mu_1}{U_{11}} \frac{1}{1 - U_{12}^2/U_{11}U_{22}} \left[1 - \frac{U_{12}\mu_2}{U_{22}\mu_1} - \frac{r^2}{R_1^2}\Big(1 - \frac{\lambda U_{12}}{U_{22}}\Big) \right],$$

$$n_2 = \frac{\mu_2}{U_{22}} \frac{1}{1 - U_{12}^2/U_{11}U_{22}} \left[1 - \frac{U_{12}\mu_1}{U_{11}\mu_2} - \frac{r^2}{R_2^2}\Big(1 - \frac{U_{12}}{\lambda U_{11}}\Big) \right],$$

where R_i are the radii of the condensate clouds for $U_{12} = 0$ ($\mu_i = \frac{1}{2}m_i\omega_i^2 R_i^2$) and λ is the constant you need to identify.

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