UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 10 June, 2005 9 to 11

PAPER 81

PHYSIOLOGICAL FLUID DYNAMICS

*Attempt **TWO** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

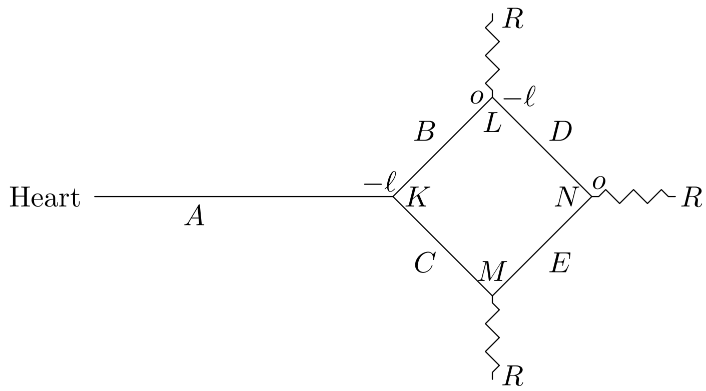
STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Answer two questions.

1 The principal arteries in the brain are modelled as follows:



In a normal subject, vessels B, C, D, E are identical, with characteristic admittance Y, wave-speed c and length l. The junctions at the points L, M, N lead directly to peripheral circulatory beds with resistance R; venous pressure may be taken to be zero. The artery leading from the heart, vessel A, has characteristic admittance $Y_A = 2Y$ and wave-speed c_A.

(i) Consider the response of the system to a single Fourier mode of the incident wave, in vessel A, for which the pressure takes the form

$$p = P_A \exp[i\omega(t - x/c_A)],$$

using standard notation. Given that the pressure in vessel B, for example, can be written

$$p = P_B E^- + P_{BR} E^+,$$

where $E^{\mp} = \exp[i\omega(t \mp x/c)]$, write down the corresponding flow-rate in vessel B. Using a similar notation for vessels C, D, E and, in each finite vessel, letting $x = 0$ at the peripheral end of the vessel (e.g. $x = 0$ at N in vessel D, $x = 0$ at L in vessel B, etc), show that the flow rate in the resistance starting from point N is $Q_N e^{i\omega t}$, where

$$Q_N = P_A R Y^2 \frac{8e^{-i\beta}}{[e^{i\beta}(2RY+1)^2 + e^{-i\beta}(2RY-1)]}$$

and $\beta = \omega l/c$.

In the limit in which $RY \gg 1$, also calculate the flow-rate, Q_M, in the resistance starting from M.

(ii) Now consider a subject in whom vessel C is obliterated by disease but nothing else is changed. Repeat the calculation of Q_N and Q_M, in the limit $RY \gg 1$, and show that $|Q_N|$, $|Q_M|$ and $|Q_L|$ are altered by factors

$$\frac{4\cos\beta}{\gamma} \qquad \frac{4\cos(\beta/2)}{\gamma\cos\beta} \qquad \frac{4\cos 2\beta}{\gamma\cos\beta},$$

respectively, where $\gamma = |3e^{4i\beta} + e^{-2i\beta}|$. Show that, for small β, all these values are greater than 1. How do you explain the increase in overall flow rate amplitude?

2 The following equations govern the flow of a fluid along a collapsible tube in which the dimensionless cross-sectional area, cross-sectionally averaged velocity and pressure are $\alpha(x,t)$, $u(x,t)$ and $p(x,t)$, respectively:

$$\alpha_t + (u\alpha)_x = 0 \qquad (1)$$

$$u_t + uu_x = -p_x - R(\alpha)u \qquad (2)$$

$$p - p_e(x) = \tilde{P}(\alpha). \qquad (3)$$

(i) Explain the significance of each term in these equations, and how they have been non-dimensionalised. What are the signs of $\tilde{P}'(\alpha)$ and $R'(\alpha)$?

(ii) Find the condition that must be satisfied by $p_e(x)$ to permit steady flow with flow rate Q and uniform cross-sectional area α_0.

(iii) For a tube in which $R(\alpha) \propto \alpha^{-n} (n > 0)$, consider small perturbations to the steady flow of (ii) in which $\alpha = \alpha_0 + \alpha'$ where $\alpha' = Ae^{i(kx-\omega t)}$ for real wave number k. Find the dispersion relation satisfied by ω, and show that ω is real if $Q/\alpha_0 = c_0/n$, where $c_0^2 = \alpha_0 \tilde{P}'(\alpha_0)$.

(iv) By considering a case in which

$$\frac{Q}{\alpha_0} = \frac{c_o}{n}(1+\delta), \quad |\delta| \ll 1,$$

or otherwise, show that the flow is unstable when $Q/\alpha_0 > c_0/n$. Show also that the growth rate of the disturbance is approximately

$$\frac{\delta R_o k^2 c_0^2}{2\,(k^2 c_0^2 + R_0^2/4),}$$

where $R_0 = R(\alpha_0)$, when $0 < \delta \ll 1$.

(v) Consider peristaltic pumping in the same tube, neglecting all fluid inertia, so that the left-hand side of equation (2) is set to zero. The external pressure is now prescribed to be

$$p_e(x,t) = \epsilon P_e \sin kX \quad \text{where} \quad X = x - ct \quad \text{and} \quad \epsilon \ll 1.$$

Seek a solution in which α, u and p are functions only of X, showing first that

$$u = c + Q/\alpha$$

for some constant Q. Then expand in powers of ϵ, so that

$$\alpha = \alpha_0 + \epsilon\alpha_1(X) + \epsilon^2\alpha_2(X) + \ldots$$

and

$$Q = Q_0 + \epsilon Q_1 + \epsilon^2 Q_2 + \ldots$$

Show that $Q_0 = -\alpha_0 c, \quad Q_1 = 0$ and

$$Q_2 = \frac{c(n+1)}{2\alpha_0} \quad P_e^2 \frac{1}{\tilde{P}_0'^2 + \left(\frac{R_0 c}{k\alpha_0}\right)^2},$$

where

$$\tilde{P}_0' = \tilde{P}'(\alpha_0).$$

3 Steady plane Poiseuille flow of average velocity $\hat{U}$ exists far upstream in an indented rigid channel whose upstream width is a. Write the dimensional Cartesian coordinates and velocity components as $(\lambda ax, ay)$ and $(\hat{U}u, \hat{U}v/\lambda)$ respectively, so x, y, u, v are dimensionless. The wall $y = 1$ is planar. The other wall is planar ($y = 0$) for $x < 0$. For $x > 0$ it is indented to $y = \epsilon F(x)$, where $F(x)$ is prescribed. The Reynolds number $R = a\hat{U}/\nu$ (ν = kinematic viscosity) is large.

Write down the dimensionless Navier-Stokes equations and boundary conditions. Show, under suitable conditions on λ and ϵ, to be explained, (a) that the perturbation to the oncoming flow can be analysed in an inviscid core and viscous boundary layers; (b) that, in the core,

$$u = U_0(y) + \epsilon A(x)U_0'(y) + O(\epsilon^2)$$
$$v = -\epsilon A'(x)U_0(y) + O(\epsilon^2)$$

where $U_0(y) = 6y(1-y)$ and $A(x)$ is an unknown function; and (c) that the boundary layers on the walls are both governed by problems of the following form:

$$U_x + V_z = 0$$
$$UU_x + VU_z = -P'(x) + U_{zz}$$
$$U = V = 0 \quad \text{on} \quad z = 0$$
$$U \sim 6[z + H(x)] \quad \text{as} \quad z \to \infty$$

where U, V, z are u, v, y suitably rescaled, $H(x)$ is a function which must be specified for each boundary layer, and P is the rescaled pressure, which is the same in the two boundary layers.

Show that, if the boundary layer problem has a unique solution, then $A = -\frac{1}{2}F$.

For the case in which $F(x) = bx^{1/3}$ for some constant b, show that the boundary-layer problem has a similarity solution, in which

$$A(x) = \bar{a}x^{\alpha}, \quad P(x) = \bar{p}x^{\sigma}, \quad U(x,z) = x^{\beta}G_{\eta}(\eta),$$

where $\eta = zx^{-\gamma}$ and α, β, γ, σ are constants which should be found, provided that $\bar{p}$ is such that the following boundary value problem has a solution:

$$G_{\eta\eta\eta} + \frac{2}{3}GG_{\eta\eta} - \frac{1}{3}G_{\eta}^2 = \frac{2}{3}\bar{p}$$

$$G(0) = G_{\eta}(0) = 0, \quad G_{\eta}(\eta) \sim 6\eta + 3b \quad \text{as} \quad \eta \to \infty.$$

4 A rigid cylindrical tube of radius a is lined by a thin layer of liquid of undisturbed thickness $h_0 \ll a$, viscosity μ and surface tension σ. The difference between the pressure in the central air core and that in the liquid layer is $\sigma(1/R_1 + 1/R_2)$, where R_1 and R_2 are the radii of curvature of the interface in the longitudinal and the transverse planes. Gravity is negligible.

(i) Assuming that $\sigma = \sigma_0$, a constant, use lubrication theory to analyse the stability of the layer to small axisymmetric perturbations in its thickness of the form $h = h_0 + h_1 e^{\beta t + ikx}$, $h_1 \ll h_0$, and show that the interface is unstable to disturbances of wave-number k such that $0 < k^2 < 1/a^2$. Show too that the most rapidly-growing disturbance has growth-rate

$$\beta_{max} = \frac{\sigma_0 h_0^3}{12\mu a^4}. \tag{1}$$

(ii) Now suppose that the interface contains insoluble surfactant molecules of concentration Γ, such that

$$\sigma = \sigma_0 - A\Gamma$$

where σ_0 and A are positive constants. The diffusion of surfactant along the interface can be neglected so its transport is entirely by advection. Repeat the analysis of part (i), with $h = h_0$ and $\Gamma = \Gamma_0$ in the undisturbed state. Show that the growth-rate β of disturbances is given by

$$\beta^2 + \beta k^2 a^2 \left[\lambda\left(k^2 a^2 - 1\right) + \alpha\right] + \frac{\alpha\lambda}{4} k^4 a^4 \left(k^2 a^2 - 1\right) = 0,$$

where

$$\lambda = \frac{h_0^3\left(\sigma_0 - A\Gamma_0\right)}{3\mu a^4}, \quad \alpha = \frac{h_0 A \Gamma_0}{\mu a^2}.$$

Deduce that

(a) the interface is unstable for $0 < k^2 a^2 < 1$;

(b) for $\frac{\alpha}{\lambda} \ll 1$, the maximum growth rate is reduced from the value given by (1), with σ_0 replaced by $\sigma_0 - A\Gamma_0$, to

$$\beta_{max} = \frac{\lambda}{4} - \frac{3\alpha}{8};$$

(c) for $\frac{\lambda}{\alpha} \ll 1$, the maximum growth rate is given by $\beta_{max} \approx \frac{\lambda}{16}$, one-quarter the value in the absence of surfactant.

(d) Explain physically why the presence of surfactant reduces the growth-rate of disturbances.

END OF PAPER