UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 2 June, 2005 1.30 to 4.30

PAPER 79

ASYMPTOTIC METHODS IN FLUID MECHANICS

*Attempt **THREE** questions.*

The questions carry equation weight.

*There are **FIVE** questions in total.*

The maximum credit available for question 1 will be divided equally between part (a) and part (b).

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) (i) Evaluate the first two terms as $m \nearrow 1$ of the elliptic integral

$$\int_0^{\pi/2} \frac{1}{(1-m^2\sin^2\theta)^{1/2}} \mathrm{d}\theta \, ,$$

given that

$$\int \frac{1}{(1+u^2)^{1/2}} \, \mathrm{d}u = \log\left(u + (1+u^2)^{1/2}\right)$$

and

$$\int \frac{1}{\sin\varphi} \, \mathrm{d}\varphi = \log\left(\tan\tfrac{1}{2}\varphi\right) \, .$$

(ii) Evaluate the first two terms as $x \to \infty$ of the integral

$$\int_0^{\infty} \frac{e^{-xu}}{(\log u)^2 + e^{-2xu}} \mathrm{d}u \, .$$

(b) (i) Briefly describe the use of the Briggs-Bers technique to determine the long-time behaviour of solutions of initial-value problems, and in particular state how to determine whether or not a system is absolutely unstable by considering its dispersion relation.

Consider the third-order equation

$$\frac{\partial \eta}{\partial t} + \alpha \frac{\partial^2 \eta}{\partial x^2} + \beta \frac{\partial^3 \eta}{\partial x^3} + \gamma \eta = 0 \, ,$$

where α, β, γ are complex constants. By considering solutions proportional to $\exp(-\mathrm{i}\omega t + \mathrm{i}kx)$, derive conditions on α and β for the system to possess a finite maximum temporal growth rate over all real k, and explain why this is important for application of the Briggs-Bers technique. Determine necessary conditions for the occurrence of absolute instability.

(ii) The nonlinear Schrodinger (NLS) equation in one dimension is

$$-\mathrm{i}\frac{\partial A}{\partial t} + \chi \frac{\partial^2 A}{\partial x^2} - \mu A + \delta A|A|^2 = 0 \, ,$$

where χ, δ and μ are real positive constants. We investigate the stability of the constant solution of NLS by writing

$$A = \left\{ \left(\frac{\mu}{\delta}\right)^{1/2} + \epsilon A_1 \right\} \exp(\mathrm{i}\epsilon\theta) \, ,$$

where $A_1 = A_1(x,t)$ and $\theta = \theta(x,t)$ are unknown real functions and $\epsilon \ll 1$. By taking terms of size $O(\epsilon)$, find a system of coupled equations for A_1 and θ. Assuming that

$$\theta, A_1 \propto \exp(-\mathrm{i}\Omega t + \mathrm{i}Kx) \, ,$$

determine the dispersion relationship between Ω and K. Comment on the stability of the constant solution of NLS.

2 By means of Watson's lemma, show that the asymptotic expansion of the integral

$$F(z) = \int_1^\infty \exp(-z^3t^3)\,\mathrm{d}t$$

for real $z \gg 1$, is

$$F(z) \sim \frac{\exp(-z^3)}{3z^3} \sum_{r=0}^{\infty} \frac{\Gamma(r+\frac{2}{3})}{\Gamma(\frac{2}{3})} \frac{1}{(-z)^{3r}}\,.$$

Hence find an asymptotic expansion, including exponentially small terms, for

$$G(z) = \int_0^1 \exp(-z^3t^3)\,\mathrm{d}t\,.$$

Now suppose that z is complex. By considering contours of steepest descent, find the asymptotic expansions of $G(z)$ for $0 \leq \arg(z) \leq 2\pi$ away from Stokes lines, which should be identified. Selected contours of constant $\Re(-z^3t^3)/|z^3|$ for complex z are plotted in figure 1 (see overleaf), together with contours of $\Im(-z^3t^3)/|z^3|$ passing through $t = 0$ and $t = 1$.

Obtain an expression for the smoothing out of the jump in the sub-dominant term at one of Stokes lines; confirm that your result is consistent with the asymptotic expansions away from that Stokes lines obtained earlier.

You may quote the following results.

(a) The Borel sum of

$$\sum_{p=0}^{\infty} \frac{\Gamma(\gamma+p)e^{\lambda}}{\lambda^{p+\gamma}}\,,$$

for real $\gamma \geq 0$ and $\Re(\lambda) > 0$, is

$$I(\lambda,\gamma) = \int_0^\infty \frac{t^{\gamma-1}\exp(\lambda(1-t))}{1-t}\,\mathrm{d}t\,,$$

where the contour of integration is assumed to pass just above the pole at $t = 1$.

(b) If $\gamma \gg 1$ and

$$\lambda \sim \gamma + \mathrm{i}\mu\gamma^{\frac{1}{2}} + \nu + \dots\,,$$

where $\mu = O(1)$ and $\nu = O(1)$ then

$$I(\lambda,\gamma) \sim i\pi\left(1 + \operatorname{erf}\left(\frac{\mu}{\sqrt{2}}\right)\right)$$

(c)

$$\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right) = \tfrac{2}{\sqrt{3}}\pi\,.$$

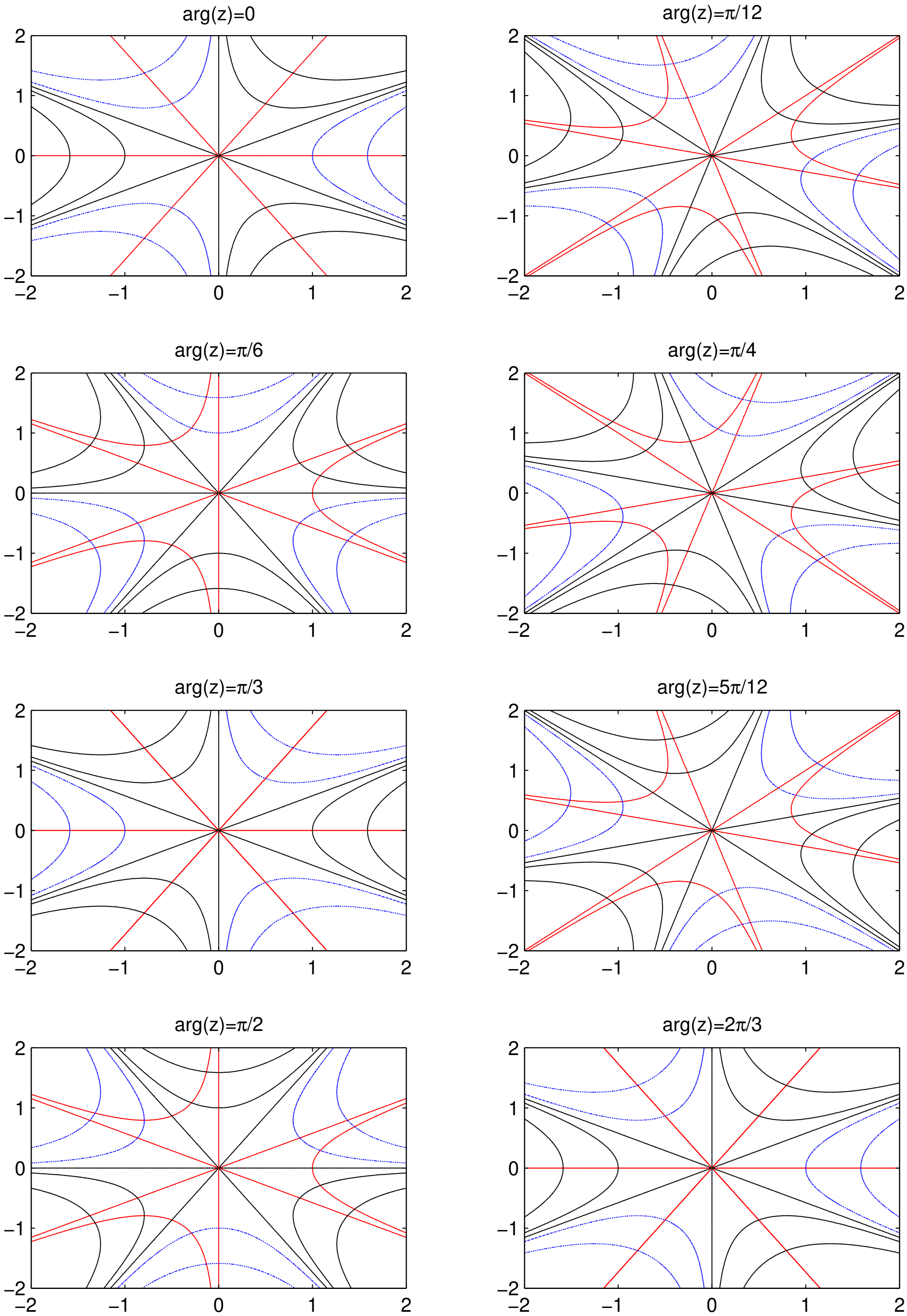

Figure 1. Contours in the complex t-plane of constant $\Re(-z^3t^3)/|z^3|$ for given complex z (black or blue depending on whether the real part is positive or strictly negative), plus contours of $\Im(-z^3t^3)/|z^3|$ passing through $t=0$ and $t=1$ (red).

3 The function $y(x)$ satisfies the nonlinear differential equation

$$\varepsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - y^{\frac{1}{2}} + \varepsilon k = 0\,,$$

where $k > 0$ is an order one constant, subject to the boundary conditions $y(0) = 0$ and $y(1) = 1$. In the limit $\varepsilon \to 0$, use the method of matched asymptotic expansions to find the leading-order solution for $0 \leq x \leq 1$. An implicit solution is acceptable as long as is it demonstrated that the solution satisfies any necessary matching conditions. Sketch the solution.

Hints.

(a) Away from the boundaries there is an outer region. Near $x = 0$ there is a single inner region, while near $x = 1$ there are two inner asymptotic regions (with the smallest region lying *between* the outer region and the asymptotic inner region that includes $x = 1$).

(b) The implicit solution to

$$\tfrac{1}{2}\left(\frac{\mathrm{d}Y}{\mathrm{d}\xi}\right)^2 - \tfrac{2}{3}Y^{\frac{3}{2}} + kY = \tfrac{1}{3}k^3$$

is given by

$$\sigma(\xi - \xi_0) = \sqrt{2k}\log\left|\frac{\sqrt{k + 2Y^{\frac{1}{2}}} - \sqrt{3k}}{\sqrt{k + 2Y^{\frac{1}{2}}} + \sqrt{3k}}\right| + \sqrt{6(k + 2Y^{\frac{1}{2}})}\,,$$

where ξ_0 is a constant and $\sigma = \pm 1$, with the sign being chosen as needed and/or appropriate. Note that

(i) if $|Y^{\frac{1}{2}} - k| \ll 1$, then $\sigma\xi \ll -1$ and

$$\ln|Y^{\frac{1}{2}} - k| \sim \frac{\sigma\xi}{\sqrt{2k}} + \dots\,;$$

(ii) if $Y^{\frac{1}{2}} \gg 1$, then $\sigma\xi \gg 1$ and

$$Y^{\frac{1}{2}} \sim \tfrac{1}{12}\xi^2 + \tfrac{1}{6}\xi\xi_0 + \dots\,.$$

(c) For $z > 0$ define

$$\zeta = \frac{\sqrt{3}}{2}\int_1^z \frac{\mathrm{d}u}{\sqrt{u^{\frac{3}{2}} + c}}\,,$$

where c is a constant. If $c = 0$, then

$$z^{\frac{1}{2}} = \tfrac{1}{12}(\zeta - \zeta_0)^2 \qquad \text{where} \qquad \zeta_0 = -2\sqrt{3}\,.$$

while if $c \neq 0$ then when $z \ll 1$

$$z \sim 2\sqrt{\frac{c}{3}}\,(\zeta - \zeta_0) \qquad \text{where} \qquad \zeta_0 = -\frac{\sqrt{3}}{2}\int_0^1 \frac{\mathrm{d}u}{\sqrt{u^{\frac{3}{2}} + c}}\,.$$

4 (a) Consider the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\epsilon f(x)\frac{\mathrm{d}x}{\mathrm{d}t}+x=0\ ,$$

subject to $x=1$ and $\mathrm{d}x/\mathrm{d}t=0$ when $t=0$, where $\epsilon\ll 1$ and $f(x)$ is a given function. Use the method of multiple scales to find the leading-order approximation to $x(t;\epsilon)$ which is uniformly valid for $t\le O(1/\epsilon)$ when

(i) $f(x)=x^2-1$,

(ii) $f(x)=\sin x$,

(iii) $f(x)=|x|$.

[Hint: in (iii) you may find it useful to write

$$|\sin\tau|\cos\tau=\sum_{n=0}^{\infty}a_n\cos(n\tau)$$

for some constants a_n.]

(b) Consider the linearised Ginsburg-Landau equation

$$\frac{\partial\eta}{\partial t}+U\frac{\partial\eta}{\partial x}-\mu\eta-\gamma\frac{\partial^2\eta}{\partial x^2}=0$$

with $\mu(X)=\mu_0-\nu X^2$ and other coefficients constant, where $X=\epsilon x$, $\epsilon\ll 1$ and we take $\nu>0$. A global mode is of the form $A=a(x)\exp(-i\omega_G t)$, with $a(x)$ bounded as $x\to\pm\infty$. Show, by writing $a(x)=\exp(Ux/2\gamma)b(\xi)$, where $\xi=\sqrt{\epsilon}cx$ for suitable c, that the global mode satisfies

$$\frac{d^2b}{d\xi^2}+(\lambda-\xi^2)b=0\ , \qquad (*)$$

for suitable λ, to be found. By seeking a solution of $(*)$ in the form $\exp(-\xi^2/2)H_n(\xi)$, where H_n is a power series, show that the solutions of (*) grow exponentially as $\xi\to\pm\infty$, unless $\lambda=(2n+1)$ for any positive integer n. Hence, determine the allowed global mode frequencies and comment on the global stability of the system.

5 (a) Consider the motion of a fluid along a straight channel aligned parallel to the x-axis, with walls $y = \pm 1$ and which is infinite in the spanwise z direction. The mean flow along the channel is $U(y)$ in the x direction, and if we assume that unsteady disturbances to the y component of velocity take the form $v(y)\exp(-\mathrm{i}\omega t + \mathrm{i}kx + \mathrm{i}lz)$, then it follows that $v(y)$ satisfies the Orr-Sommerfeld equation

$$(U-c)\left(\frac{d^2}{dy^2} - k^2 - l^2\right)v - \frac{d^2U}{dy^2}v - \frac{1}{\mathrm{i}kR}\left(\frac{d^2}{dy^2} - k^2 - l^2\right)^2 v = 0\ , \qquad (*)$$

where $c = \omega/k$ and R is the Reynolds number. The boundary conditions are that v and dv/dy vanish on $y = \pm 1$.

(i) In the inviscid case $R = \infty$, prove Rayleigh's Theorem that a necessary condition for the existence of temporal instability is that $U(y)$ possesses an inflection point in the channel.

(ii) Show that the solution of the Orr-Sommerfeld equation (*) for given values of k, l, R can be related to the solution of the same equation but with $l = 0$ and with k, R replaced by new values k', R' (to be determined). Use this to explain why the Orr-Sommerfeld equation first becomes unstable to two-dimensional disturbances as the Reynolds number is increased.

(iii) The Squire equation is

$$(-\mathrm{i}\omega + \mathrm{i}kU)\eta - \frac{1}{R}\left(\frac{d^2}{dy^2} - k^2 - l^2\right)\eta = 0\ ,$$

subject to $\eta(\pm 1) = 0$. Show that solutions of the Squire equation are temporally stable. [Hint: Multiply the Squire equation by the complex conjugate of $\eta(y)$ and integrate over $-1 \le y \le 1$.]

(b) Consider the first-order ordinary differential equation

$$\frac{d\mathbf{q}}{dt} = A\mathbf{q}\ ,$$

where $\mathbf{q}(t)$ is an N-dimensional vector and A is a constant $N \times N$ matrix. The optimal growth, $G(t)$, is defined to be the maximum value of

$$\frac{\| \mathbf{q}(t) \|}{\| \mathbf{q}(0) \|}$$

maximised over all $\mathbf{q}(0) \neq 0$. Determine lower and upper bounds on $G(t)$ in terms of properties of the matrix A, and calculate these bounds explicitly in the case

$$M = \begin{pmatrix} -\frac{1}{R} & 0 \\ 1 & -\frac{2}{R} \end{pmatrix}\ . \qquad (**)$$

For the choice of M given in (**), show that

$$\exp(Mt) = \begin{pmatrix} \exp(-t/R) & 0 \\ -[\exp(-2t/R) - \exp(-t/R)]R & \exp(-2t/R) \end{pmatrix}\ .$$

Determine the initial conditions which lead to optimal growth at time t when (i) $t \gg R$, (ii)$t \ll R$.

END OF PAPER