UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 3 June, 2005 1.30 to 4.30

PAPER 78

NONLINEAR CONTINUUM MECHANICS

*Attempt **FOUR** questions.*

*There are **SIX** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Prove the polar decomposition theorem, that any $n \times n$ matrix $\mathbf{F}$ with positive determinant can be decomposed so that $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$, where $\mathbf{U}$ and $\mathbf{V}$ are symmetric with positive eigenvalues and $\mathbf{R}$ is proper orthogonal.

Define a strain measure $\mathbf{E}^f$ associated with a function f, stating what properties f is required to have. Demonstrate that $\mathbf{E}^{(1)} = \mathbf{U} - \mathbf{I}$ is such a measure.

State what is meant by the stress measure $\mathbf{T}^f$ conjugate to the strain measure $\mathbf{E}^f$. Express the "symmetric Biot" stress $\mathbf{T}^{(1)}$ conjugate to $\mathbf{E}^{(1)}$ in terms of the nominal stress $\mathbf{P}$. [*You are reminded that the rate of working of stress per unit reference volume is* $P_{Ii}\dot{F}_{iI}$.]

2 A long hollow cylinder which, when undeformed, has inner radius a_0 and outer radius b_0, is composed of isotropic incompressible Green-elastic material with energy function $W(\lambda_1, \lambda_2, \lambda_3)$ per unit reference volume, where λ_r $(r = 1, 2, 3)$ denote the principal stretches. The cylinder is subjected to a uniform axial stretch λ_z, and to uniform radial inflation so that the inner radius becomes a. This deformation is achieved by internal pressure p and resultant axial force N.

Employing the usual notation that $R \to r$ while $X_3 \to x_3 = \lambda_z X_3$, show that the circumferential stretch at radius R is

$$\lambda \equiv \frac{r}{R} = \left[1 + \frac{\lambda_z a^2 - a_0^2}{R^2}\right]^{1/2} \lambda_z^{-1/2}.$$

By equating the rate of working of the applied loads to the rate of increase of stored energy as a and λ_z are varied, show that

$$p(a, \lambda_z) = \lambda_z^{-1} \int_{a_0}^{b_0} \frac{\partial \hat{W}}{\partial \lambda} \frac{dR}{\lambda R},$$

where $\hat{W}(\lambda, \lambda_z) = W(1/(\lambda\lambda_z), \lambda, \lambda_z)$. Also give the corresponding formula for the axial load N.

For the particular energy function

$$W(\lambda_1, \lambda_2, \lambda_3) = \frac{1}{2}\mu_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) - \frac{1}{2}\mu_2(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} - 3),$$

show that

$$p(a, \lambda_z) = (\mu_1 \lambda_z^{-2} - \mu_2)\left\{\lambda_z \ln(\lambda_a/\lambda_b) - \frac{1}{2}(\lambda_a^{-2} - \lambda_b^{-2})\right\},$$

where $\lambda_a = a/a_0$ and λ_b is the circumferential stretch at the outer surface $(R = b_0)$. [*You may find it convenient to transform the variable of integration from R to λ.*]

3 Derive, from global statements of the balance of linear momentum and energy, and the entropy inequality, all relative to the initial configuration, the equations

$$\begin{aligned}
\rho_0 \ddot{x}_i &= P_{Ii,I} + \rho_0 g_i, \\
\rho_0 \dot{u} &= P_{Ii}\dot{F}_{iI} + \rho_0 r - q^0_{I,I}, \\
\rho_0 \dot{\eta} &= \frac{\rho_0 r - q^0_{I,I}}{\theta} + \frac{\theta_{,I} q^0_I}{\theta^2} + \gamma; \quad \gamma \geqslant 0.
\end{aligned}$$

Here, mass density is ρ_0, internal energy per unit mass is u, θ is temperature, η is entropy per unit mass, q^0_I are components of the "nominal" heat flux vector, r is rate of heat supply per unit mass, g_i is body force per unit mass, P_{Ii} is nominal stress, $F_{iI} = \partial x_i/\partial X_I$ and all fields are assumed to be smooth.

Now suppose that the fields are smooth, except possibly across a surface which maps back to a surface $S(t)$ in the reference configuration, which moves with normal velocity V in that configuration. Deduce from the linear momentum balance the jump condition

$$\rho_0 V[\dot{x}_i] = n^0_I [P_{Ii}],$$

where n^0_I are the components of the normal to $S(t)$. Give also the jump conditions corresponding to the energy balance and the entropy inequality. [*You will probably be able to recognise what these should be, without the need to mirror in detail the argument given for the momentum balance.*]

4 The Jaumann or co-rotational derivative of a second-order tensor $\mathbf{T}$ is defined as

$$\frac{\mathcal{D}\mathbf{T}}{\mathcal{D}t} = \dot{\mathbf{T}} - \mathbf{W}\mathbf{T} + \mathbf{T}\mathbf{W},$$

where $\mathbf{W}$ is the material spin, $\mathbf{W} = (\mathbf{L} - \mathbf{L}^T)/2$ in the usual notation. Show that, if the rotation $\mathbf{Q}$ is such that $\dot{\mathbf{Q}}\mathbf{Q}^T = \mathbf{W}$, then

$$\frac{d}{dt}(\mathbf{Q}^T\mathbf{T}\mathbf{Q}) = \mathbf{Q}^T\frac{\mathcal{D}\mathbf{T}}{\mathcal{D}t}\mathbf{Q}.$$

The "co-rotational Jeffreys" constitutive relation for incompressible fluid is

$$\frac{\mathcal{D}\boldsymbol{\sigma}^d}{\mathcal{D}t} + \frac{1}{\tau}\boldsymbol{\sigma}^d = 2\mu_0\frac{\mathcal{D}\mathbf{D}}{\mathcal{D}t} + \frac{2\mu_1}{\tau}\mathbf{D},$$

together with $\sigma_{ij} = \sigma^d_{ij} - p\delta_{ij}$, $D_{k,k} = 0$, where $\mathbf{D}$ is the strain-rate, $(\mathbf{L} + \mathbf{L}^T)/2$. Show that

$$\boldsymbol{\sigma}^d(t) = 2\mu_0\mathbf{D}(t) + \frac{2(\mu_1 - \mu_0)}{\tau}\int_{-\infty}^{t} e^{-(t-t')/\tau}\mathbf{Q}(t)\mathbf{Q}^T(t')\mathbf{D}(t')\mathbf{Q}(t')\mathbf{Q}^T(t)\,dt'.$$

For the simple shear deformation

$$\mathbf{F}(t) = \begin{pmatrix} 1 & \gamma(t) \\ 0 & 1 \end{pmatrix}$$

(the irrelevant 3-components being suppressed), show that this relation implies

$$\boldsymbol{\sigma}^d(t) = \mu_0\dot{\gamma}(t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{(\mu_1 - \mu_0)}{\tau}\int_{-\infty}^{t} e^{-(t-t')/\tau}\dot{\gamma}(t')\begin{pmatrix} -\sin 2(\theta - \theta') & \cos 2(\theta - \theta') \\ \cos 2(\theta - \theta') & \sin 2(\theta - \theta') \end{pmatrix} dt',$$

where $\theta(t)$ denotes the angle between the fixed Eulerian frame and a frame rotating with angular speed $\dot{\gamma}(t)/2$, and the shorthand $\theta = \theta(t)$, $\theta' = \theta(t')$ has been employed.

In the special case that $\dot{\gamma}$ is constant for all time, show that

$$\sigma^d_{12} = \left(\frac{\mu_1 + \dot{\gamma}^2\tau^2\mu_0}{1 + \dot{\gamma}^2\tau^2}\right)\dot{\gamma}$$

and find corresponding formulae for σ^d_{11} and σ^d_{22}.
[*The result* $\int_0^\infty e^{-s/\tau}(\cos\dot{\gamma}s + i\sin\dot{\gamma}s)ds = \tau/(1 - i\dot{\gamma}\tau)$ *can be used.*]

5 (a) Fluid with constitutive relation

$$\sigma_{ij} = \frac{\partial \Omega(\mathbf{D})}{\partial D_{ij}},$$

where the function Ω is convex, occupies a domain $\mathcal{D}$ with boundary S. It is subjected to body force with components ρg_i per unit volume. Components of velocity v_i are prescribed so that $v_i = v_i^0$ over a part S_v of the boundary, while components of traction, $\sigma_{ij} n_j = t_i^0$, are prescribed over the complementary part S_t. Assuming that the deformation is slow so that inertia can be ignored, show that the actual velocity field $\mathbf{v}$ minimises the functional

$$\mathcal{F}(\mathbf{v}') = \int_{\mathcal{D}} [\Omega(\mathbf{D}') - \rho g_i v_i'] dx - \int_{S_t} t_i^0 v_i' dS,$$

over fields $\mathbf{v}'$ that satisfy the restriction $\mathbf{v}' = \mathbf{v}^0$ on S_v. [*The property* $\Omega(\mathbf{D}') - \Omega(\mathbf{D}) \geqslant (D'_{ij} - D_{ij}) \partial \Omega(\mathbf{D}) / \partial D_{ij}$ *of the convex function* Ω *may be assumed.*]

(b) This part concerns the limiting case of a Bingham fluid, for which

$$\Omega(\mathbf{D}) = \tau_0 \dot{\gamma} + \frac{1}{2} \mu_0 \dot{\gamma}^2,$$

where $\dot{\gamma} = (2 D_{ij} D_{ij})^{1/2}$. The fluid is incompressible, so $D_{kk} = 0$ and the stress is determined only up to an unknown pressure. Show that, if $\dot{\gamma} > 0$,

$$\sigma'_{ij} = \frac{\partial \Omega}{\partial D_{ij}} = 2 \left(\frac{\tau_0}{\dot{\gamma}} + \mu_0 \right) D_{ij},$$

where σ'_{ij} is the deviatoric stress. State the restriction on σ'_{ij} when $\dot{\gamma} = 0$.

Flow between fixed plates, situated at $x_2 = \pm h$, is driven by a uniform pressure gradient G parallel to the x_1-axis. Assuming that the only non-zero velocity component is v_1 and that this is an odd function of x_2 only, and that the stress component σ_{12} is continuous and an odd function of x_2 only, show that $\sigma_{12} = -G x_2$, and hence that no flow is possible unless $G > \tau_0 / h$. Find the velocity profile when this condition is met.

6 Non-hardening, rigid-perfectly plastic material is subjected to "antiplane" deformation so that the only non-zero component of velocity is v_3 and the only non-zero components of stress are σ_{13}, σ_{23}, all functions of (x_1, x_2) only. No body-force is applied. The material has yield condition (under such deformation)

$$f(\sigma_{13}, \sigma_{23}) = k,$$

and it conforms to the associated flow law $\dot{\varepsilon}_{i3} = \dot{\lambda}\partial f/\partial\sigma_{i3}$, $i = 1, 2$.

By considering $d\sigma_{13}/ds$ along a curve defined by $x_1 = x_1(s)$, $x_2 = x_2(s)$, in conjunction with the relevant equation of equilibrium and the derivative with respect to x_2 of the yield criterion, show that σ_{13} is constant along the characteristic curve, defined so that $dx_1/ds = -\alpha\partial f/\partial\sigma_{23}$, $dx_2/ds = \alpha\partial f/\partial\sigma_{13}$. Show similarly that σ_{23} and v_3 are constant along any characteristic curve.

Apply this result to obtain the form of the solution close to the tip of a crack: the crack occupies $x_1 < 0$, $x_2 = 0$ and its surfaces are traction-free. The required field is a "centred fan", centred at the crack tip. Explain why it is bounded by a line that makes an angle ϕ_b with the x_1-axis, where

$$\tan\phi_b = -\frac{\partial f/\partial\sigma_{23}}{\partial f/\partial\sigma_{13}},$$

evaluated where $\sigma_{23} = 0$.

In the special case

$$f(\sigma_{13}, \sigma_{23}) = \frac{\sigma_{13}^2}{A^2} + \frac{\sigma_{23}^2}{B^2} = 1,$$

show that, in the centred fan region,

$$\sigma_{13} = \frac{-A^2\tan\phi}{(B^2 + A^2\tan^2\phi)^{1/2}}, \quad \sigma_{23} = \frac{B^2}{(B^2 + A^2\tan^2\phi)^{1/2}},$$

where r, ϕ are polar coordinates based on the crack tip.

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