UNIVERSITY OF
CAMBRIDGE

## MATHEMATICAL TRIPOS Part III

Tuesday 7 June, 2005 9 to 12

# PAPER 77

# SLOW VISCOUS FLOW

*Attempt at most **THREE** questions.*

*There are **FOUR** questions in total.*

*The questions carry equal weight.*

*Substantially complete answers will be viewed more favourably than fragments.*

*A Distinction mark can be gained by substantially complete answers to two questions.*

***STATIONERY REQUIREMENTS***
*Cover sheet*
*Treasury Tag*
*Script paper*

***SPECIAL REQUIREMENTS***
*None*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

*The Papkovich–Neuber representation of Stokes flow is*

$$\mathbf{u} = \boldsymbol{\nabla}(\mathbf{x} \cdot \boldsymbol{\Phi} + \chi) - 2\boldsymbol{\Phi}, \quad p = 2\mu\boldsymbol{\nabla} \cdot \boldsymbol{\Phi}, \quad \text{where } \nabla^2\chi = 0 \text{ and } \nabla^2\boldsymbol{\Phi} = \mathbf{0}.$$

**1** Let $\mathbf{u}$ be a Stokes flow (with no body force) in a region $V$ with bounding surface $\partial V$ and outward normal $\mathbf{n}$. Given that the local rate of viscous dissipation is $\boldsymbol{\sigma} : \mathbf{e}$, where $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{e}$ the strain-rate tensor, show that the total dissipation is given by

$$D = \int_{\partial V} \mathbf{u} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \, dS.$$

Let $\mathbf{u}_0$ be the Stokes flow in $V_0$ due to a specified velocity $\mathbf{U}_0(\mathbf{x})$ on $\partial V_0$, and let $\mathbf{u}_0 + \mathbf{u}'$ be the Stokes flow, with stress tensor $\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}'$, produced by adding a rigid, force-free, couple-free particle to the flow while maintaining the velocity equal to $\mathbf{U}_0(\mathbf{x})$ on $\partial V_0$. Show that the increase $D'$ in dissipation due to the presence of the particle is given by

$$D' = \int_{\partial V_0} \mathbf{U}_0 \cdot \boldsymbol{\sigma}' \cdot \mathbf{n} \, dS$$

and, using the Reciprocal Theorem or otherwise, show further that

$$D' = \int_A (\mathbf{u}_0 \cdot \boldsymbol{\sigma}' - \mathbf{u}' \cdot \boldsymbol{\sigma}_0) \cdot (-\mathbf{n}) \, dS,$$

where $A$ is the surface of the particle and $-\mathbf{n}$ is its outward normal.

A force-free couple-free rigid sphere of radius $a$ is placed at the origin in an *unbounded* strain flow with uniform strain rate $\mathbf{E}$. Find the perturbation to the flow arising from the presence of the sphere. Given that the total stress on the surface of the sphere is

$$\boldsymbol{\sigma}_0 + \boldsymbol{\sigma}' = 5\mu\big\{(\mathbf{n} \cdot \mathbf{E} \cdot \mathbf{n})(\mathbf{I} - 2\mathbf{nn}) + (\mathbf{E} \cdot \mathbf{n})\mathbf{n} + \mathbf{n}(\mathbf{E} \cdot \mathbf{n})\big\},$$

calculate the increase in dissipation due to the presence of the sphere.

[*You may assume that the difference between the unbounded situation and the case* $\mathbf{U}_0 = \mathbf{E} \cdot \mathbf{x}$ *on* $r = R$, *where* $R \gg a$, *is negligible near the sphere.*]

A volume fraction $\phi$ of such spheres is now distributed throughout the straining flow, where $\phi \ll 1$ so that interaction between the spheres can be neglected. Calculate the number of spheres per unit volume and deduce that the average increase in dissipation per unit volume is such that the fluid appears to have an effective viscosity

$$\mu_{\text{eff}} = \mu(1 + \tfrac{5}{2}\phi).$$

**2** (a) Using Papkovich–Neuber potentials $\chi$ and $\mathbf{\Phi} = (0, \phi)$, find the two-dimensional Stokes flow $\mathbf{u} = (u, v)$ in the half-space $y \geqslant 0$ that satisfies $\mathbf{u} \to \mathbf{0}$ as $y \to \infty$ and $\mathbf{u} = (U\mathrm{e}^{ikx}, 0)$ on $y = 0$, where $U$ and $k$ are positive constants. Show that $\sigma_{xy} = -2\mu k U \mathrm{e}^{ikx}$ and $\sigma_{yy} = 0$ on $y = 0$.

[*In Part (b) you may assume that a similar calculation for* $\mathbf{u} = (0, V\mathrm{e}^{ikx})$ *on* $y = 0$ *gives* $\sigma_{xy} = 0$ *and* $\sigma_{yy} = -2\mu k V \mathrm{e}^{ikx}$.]

(b) A thin layer of fluid of viscosity $\mu$ and density $\rho$ in $-h(x,t) < z < h(x,t)$ is sandwiched between semi-infinite layers of fluid of viscosity $\lambda\mu$ and density $\rho - \Delta\rho$ in $z > h$ and of viscosity $\lambda\mu$ and density $\rho + \Delta\rho$ in $z < -h$. The $z$-direction is vertical, surface tension and inertia are negligible, and you may assume that the symmetry about $z = 0$ is maintained as the layer relaxes towards a uniform thickness under gravity.

Consider long-wavelength variations ($|\partial h/\partial x| \ll 1$), and suppose that $\lambda$ is sufficiently small that the horizontal velocity $u$ in the layer is approximately uniform ($|\boldsymbol{\nabla} u| \ll u/h$). Starting from $u = u(x,t)$, use the vertical stress balance and the horizontal force balance to derive the equation for extensional flow in the layer,

$$4\mu \frac{\partial}{\partial x}\left(h \frac{\partial u}{\partial x}\right) = h\,\frac{\partial}{\partial x}(\Delta\rho\, g h - \sigma^+_{zz}) - \sigma^+_{xz},$$

where $\boldsymbol{\sigma}^+$ is the modified stress in the fluid just above $z = h$. Write down the corresponding equation of conservation of mass for the layer.

Consider small perturbations to a uniform thickness $h = h_0$ with amplitude proportional to $\exp(ikx - st)$, where $\kappa \equiv kh_0 \ll 1$. Assume that $\sigma^+_{zz}$ is negligible and that $\sigma^+_{xz}$ is given by the analysis in Part (a) with $y = z - h$ and viscosity $\lambda\mu$. Show from the linearised equations that

$$s = \frac{\Delta\rho\, g h_0}{2\mu}\frac{\kappa}{\lambda + 2\kappa}.$$

Describe the dominant source of viscous resistance in each of the cases $\lambda \ll \kappa$ and $\kappa \ll \lambda$.

(c) A lengthy calculation using the unapproximated Stokes equations in all three layers shows that the linearised decay rate is given for arbitrary $\lambda$ by

$$s = \frac{\Delta\rho\, g h_0}{2\mu}\frac{\kappa + \frac{2}{3}\lambda\kappa^2 + O(\kappa^3, \lambda\kappa^4)}{\lambda + 2\kappa + O(\kappa^3)},$$

when $\kappa \ll 1$. Find the leading-order behaviour of $s$ when $\lambda \gg \kappa^{-1} \gg 1$, and suggest the dominant source of viscous resistance in this case.

**3** A rigid cylindrical tube, radius $a$, contains fluid of viscosity $\mu$ and a force-free, couple-free rigid sphere with radius $b$ and centre at distance $c$ (with $b + c < a$) from the axis of the tube. Far ahead of and behind the sphere there is uniform Poiseuille flow. Explain why

(i) $c$ is constant as the sphere is carried along by the flow,

(ii) $a \displaystyle\int_{x_1}^{x_2} \int_0^{2\pi} \sigma_{rx}(a, \theta, x)\, d\theta\, dx = \pi a^2[p(x_2) - p(x_1)]$,

where $(r, \theta, x)$ are cylindrical polar coordinates aligned with the tube, and $x_1$ and $x_2$ are two positions far from the sphere.

Consider the case $b = (1 - \epsilon)a$ and $c = \epsilon\lambda a$, where $\epsilon \ll 1$ and $0 \leqslant \lambda < 1$. Work in the frame moving with the sphere. Let the walls of the tube have velocity $-U$, and assume that the angular velocity of the sphere is negligible. The coordinates are chosen such that the width of the narrow gap between the sphere and the tube can be approximated by

$$h(\theta, x) = h_0(\theta) + x^2/(2a), \quad \text{where} \quad h_0 = \epsilon(1 + \lambda\cos\theta)a.$$

Use scaling arguments to estimate the typical magnitudes of (a) the pressure gradient, the pressure and the shear stress in the narrow gap and (b) the pressure gradient and the shear stress ahead of and behind the sphere.

Show that in the gap

$$\left.\frac{\sigma_{xy}}{\mu}\right|_{y=0} = \frac{4U}{h} + \frac{6q}{h^2}, \quad \text{where} \quad q = -\frac{h^3}{12\mu}\frac{\partial p}{\partial x} - \frac{Uh}{2} \quad \text{and} \quad y = a - r,$$

and find a similar expression for $\sigma_{xy}$ on $y = h$. Explain carefully why $q$ is approximately independent of $x$.

By considering (ii) at $O(\epsilon^{-3/2})$, show that $q = -\frac{2}{3}Uh_0(\theta)$. Deduce that the pressure gradient far from the sphere is approximately

$$8\mu U(1 - \tfrac{4}{3}\epsilon)/a^2.$$

[*You may assume that if* $I_n \equiv \displaystyle\int_{-\infty}^{\infty} \frac{d\xi}{(1 + \xi^2)^n}$ *then* $I_1 = \pi$, $I_2 = \dfrac{\pi}{2}$ *and* $I_3 = \dfrac{3\pi}{8}$. *You may also assume that the volume flux in Poiseuille flow is* $(\pi a^4/8\mu)\partial p/\partial x$.]

By considering (ii) at $O(\epsilon^{-1/2})$, show further that the leading-order pressure drop across the sphere is

$$\sqrt{\frac{2}{\epsilon}}\,\frac{2\mu U}{a}\int_0^{2\pi} \frac{d\theta}{\sqrt{1 + \lambda\cos\theta}}.$$

Show that $\displaystyle\int \sigma_{xy}|_{y=h} dx = 0$ and comment on the significance of this result.

**4** Fluid of viscosity $\mu$ and density $\rho + \Delta\rho$ spreads as a gravity current beneath fluid of density $\rho$ and over a rigid horizontal surface $z = 0$. A constant uniform shear stress $\tau$ is exerted on the upper surface of the gravity current by some mechanism (such as an imposed background flow in the upper fluid). Surface tension is negligible.

Assuming that the gravity current can be described by lubrication theory, show that its thickness $h(x, y, t)$ obeys

$$\frac{\partial h}{\partial t} + \frac{\tau}{2\mu}\frac{\partial h^2}{\partial x} = \frac{\Delta\rho\, g}{3\mu}\boldsymbol{\nabla}\cdot\left(h^3\boldsymbol{\nabla}h\right),$$

where $x$ and $y$ are the horizontal coordinates parallel and perpendicular to the shear stress $\tau$. If $h$ has typical magnitude $H$ and varies on a horizontal lengthscale $L$ what dimensionless groups must be small for the approximations of lubrication theory to hold?

Consider the case of steady flow from a point source at the origin of constant volume flux $Q$. At large distances $x$ downstream, the thickness $h(x, y)$ and cross-stream width $2y_N(x)$ of the current satisfy $h \ll y_N \ll x$. By making suitable approximations and scaling estimates, show that $y_N \propto x^{1/3}$. Hence find a similarity solution for $h(x, y)$ and determine the corresponding $y_N(x)$.

Sufficiently close to the source, the similarity solution does not apply since the lengthscales in the $x$ and $y$ directions are comparable. Use scaling arguments to estimate the distance that the current spreads upstream of the source, explaining the balance that determines this distance.

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