UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 9 June, 2005 1.30 to 3.30

PAPER 76

DYNAMO THEORY

*Attempt **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) A uniform magnetic field $\mathbf{B}_0$ permeates a fluid of magnetic diffusivity η and is subjected to the effect of the differential rotation associated with the velocity field $\mathbf{u} = \boldsymbol{\omega}(\mathbf{x}) \times \mathbf{x}$, where $\boldsymbol{\omega}(\mathbf{x})$ is everywhere parallel to $\mathbf{B}_0$. Using cylindrical polar coordinates (s, ϕ, z), with Oz parallel to $\mathbf{B}_0$, so that $\mathbf{B}_0 = (0, 0, B_0)$ and $\mathbf{u} = (0, s\omega(s, z), 0)$, show that a toroidal magnetic field $(0, B(s, z, t), 0)$ is generated, where B satisfies the equation

$$\frac{\partial B}{\partial t} = s(\mathbf{B}_0 \cdot \nabla)\omega + \eta \left(\nabla^2 - \frac{1}{s^2} \right) B \,. \qquad (*)$$

(ii) Now adopt spherical polar coordinates (r, θ, ϕ) and suppose that $\omega = \omega(r)$ where $\omega(r)$ is a smooth function of r, finite at $r = 0$ and falling to zero more rapidly than r^{-6} as $r \to \infty$. Verify that equation $(*)$ admits a steady solution of the form

$$B = -\frac{B_0}{\eta} \frac{\sin\theta \cos\theta}{r^3} \int_0^r x^4 \omega(x) dx \,.$$

(iii) Estimate the time it would take to establish this steady solution in a fluid of very high conductivity, starting from an initial condition $B = 0$ at $t = 0$.

(iv) Explain in physical terms the effect of the same differential rotation if $\mathbf{B}_0$ is in the plane of the velocity field $\mathbf{u}$ (rather than perpendicular to this plane).

2 (i) Explain the principles of *Mean Field Electrodynamics* leading to the equation

$$\boldsymbol{\mathcal{E}} = \alpha \mathbf{B} - \beta \nabla \times \mathbf{B} + \ldots ,$$

where $\boldsymbol{\mathcal{E}}$ is the mean electromotive force associated with a homogeneous isotropic field of turbulence, and $\mathbf{B}$ is the mean magnetic field.

(ii) Using the first-order smoothing approximation (which should be justified), obtain a relationship between the parameter α and the mean helicity of the turbulence.

(iii) Obtain a criterion for dynamo action resulting solely from the α-effect.

(iv) Discuss the application of this theory to the problem of the geodynamo.

3 Consider an interface dynamo with a magnetic field

$$\mathbf{B}(x,z,t) = \begin{cases} (-\partial A/\partial z, B, \partial A/\partial x) & z > 0 \\ (-\partial a/\partial z, b, \partial a/\partial x) & z < 0 \end{cases}$$

referred to Cartesian co-ordinates, such that the components of **B** satisfy the scalar equation

$$\frac{\partial B}{\partial t} - \eta \nabla^2 B = 0 \quad , \quad \frac{\partial A}{\partial t} - \eta \nabla^2 A = \alpha B$$

for $z > 0$, and the equation

$$\frac{\partial a}{\partial t} - \eta \nabla^2 a = 0 \quad , \quad \frac{\partial b}{\partial t} - \eta \nabla^2 b = V \frac{\partial a}{\partial x}$$

for $z < 0$, where η, α, V are positive constants. Explain the significance of the various terms in these equations and obtain the four conditions relating A, a and B, b at $z = 0$, given that the mean velocity is continuous. [Hint: remember the electric field.]

Now consider surface dynamo waves with

$$(A, B) = (A_0 + Cz, B_0)e^{st - \Lambda z} e^{i(kx - Lz - \omega t)},$$
$$(a, b) = (a_0, b_0 + cz)e^{st + \lambda z} e^{i(kx + lz - \omega t)},$$

where Λ, λ are positive constants. Show that the amplitude of these waves grows exponentially with time if $D > 32$, where the dynamo number

$$D = \frac{V\alpha}{\eta^2 k^3} .$$

How can this simple model be related to cyclic activity in stars like the Sun?

END OF PAPER