UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 10 June, 2005 1.30 to 4.30

PAPER 75

MAGNETOHYDRODYNAMICS AND TURBULENCE

*Attempt **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

You may quote the attached forumulae from the NRL Plasma Formulary.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
Extract from NRL Plasma Formulary

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Kinetic Alfvén Waves

In Electron (Hall) MHD, one assumes that magnetic field lines are frozen into the electron flow velocity $\mathbf{u}_e$, which can be expressed in terms of the ion velocity $\mathbf{u}$ and the current density $\mathbf{j}$ by means of the formula $\mathbf{j} = en(\mathbf{u} - \mathbf{u}_e)$, where e is the electron charge and $n = n_i = n_e$ is the ion/electron density.

1. Write a closed system of equations for $\mathbf{B}$ and $\mathbf{u}$, assuming incompressibility and neglecting viscosity and Ohmic diffusion.
2. Consider the equilibrium with a straight uniform magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}} = \text{const}$. Derive the dispersion relation for waves in this system. You will find the following definitions useful: $v_A = B_0/(4\pi n m_i)^{1/2}$ is the Alfvén speed with m_i the ion mass; $\omega_{pi} = (4\pi e^2 n/m_i)^{1/2}$ is the plasma frequency.
3. Obtain an explicit formula for the frequency ω from your dispersion relation. Under what conditions do you recover Alfvén waves?
4. The quantity c/ω_{pi} is called the plasma skin depth. Assume $ck/\omega_{pi} \gg 1$ (k is the absolute value of the wave vector) and find the corresponding limiting form of the dispersion relation. The waves you have obtained are called *kinetic Alfvén waves.*
5. If you assumed from the outset that the characteristic scale of all fields $l \ll c/\omega_{pi}$ and the charateristic time is $\tau \ll l/v_A$, how would the evolution equations for $\mathbf{u}$ and $\mathbf{B}$ simplify? Based on these simplified equations, argue, without rederiving the dispersion relation, that the kinetic Alfvén wave could have been obtained using only the equation for the magnetic field, without recourse to the momentum equation.

2 Stability of the θ Pinch

Consider a cylindrically symmetric static equilibrium with magnetic field $\mathbf{B}_0 = B_0(r)\hat{\mathbf{z}}$, pressure $p_0 = p_0(r)$, and no gravity (the θ pinch).

1. What form does the force balance take for this equilibrium?
2. Consider displacements in the form $\boldsymbol{\xi} = \hat{\boldsymbol{\xi}}(r)\exp(im\theta + ikz)$. The general expression for the perturbation of the potential energy is then

$$\delta W = 2\pi L_z \int dr\, r \left[\gamma p_0 |\nabla\cdot\boldsymbol{\xi}|^2 + \frac{|\mathbf{Q}|^2}{4\pi} + (\nabla\cdot\boldsymbol{\xi}^*)\boldsymbol{\xi}\cdot\nabla p_0 + \frac{\mathbf{j}_0\cdot(\boldsymbol{\xi}^*\times\mathbf{Q})}{c}\right],$$

   where $\mathbf{j}_0$ is the unperturbed current density and $\mathbf{Q} = \delta\mathbf{B} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}_0)$. Show that the θ pinch is always stable.

3 Advection of Magnetic Field in Ideal MHD

If Ohmic diffusion is ignored, the randomly advected magnetic field $\tilde{\mathbf{B}}(t)$ satifies the following equation (in the Lagrangian frame):

$$\frac{\partial \tilde{B}^i}{\partial t} = \sigma^i_m \tilde{B}^m,$$

where $\sigma^i_m(t) = \partial u^i / \partial x^m$ is the gradient of the velocity field. Note that there is no explicit dependence on the space variable anywhere. Take the velocity field to be a Gaussian random field white in time, three-dimensional and isotropic, and

$$\langle \sigma^i_m(t) \sigma^j_n(t') \rangle = \delta(t-t') \kappa_2 T^{ij}_{mn}, \quad T^{ij}_{mn} = \left[\delta^{ij} \delta_{mn} - \frac{1}{4} (\delta^i_m \delta^j_n + \delta^i_n \delta^j_m) \right].$$

1. Let $\tilde{P}(\mathbf{B}, t) = \delta(\mathbf{B} - \tilde{\mathbf{B}}(t))$. Define the PDF of the magnetic field $P(\mathbf{B}, t) = \langle \tilde{P}(\mathbf{B}, t) \rangle$. Derive a closed equation for this PDF using the Furutsu-Novikov formula
$$\langle \sigma^i_m(t) \tilde{P}(t) \rangle = \int dt' \langle \sigma^i_m(t) \sigma^j_n(t') \rangle \left\langle \frac{\delta \tilde{P}(t)}{\delta \sigma^j_n(t')} \right\rangle.$$

2. Because of isotropy, P only depends on the absolute value $B = |\mathbf{B}|$, so the normalisation rule is
$$1 = \int d^3\mathbf{B} P(B) = 4\pi \int dB\, B^2 P(B).$$
Define $F(B) = 4\pi B^2 P(B)$ and derive an equation for the evolution of $F(B)$. The result should be
$$\frac{\partial F}{\partial t} = \frac{\gamma}{5} \frac{\partial}{\partial B} (B^2 \frac{\partial F}{\partial B} - 2BF), \tag{1}$$
where $\gamma = (5/4)\kappa_2$.

3. What is the expression for magnetic energy $\langle B^2 \rangle(t)/8\pi$ in terms of the function $F(B, t)$? Derive an equation for $\langle B^2 \rangle(t)$ and show that $\langle B^2 \rangle(t)$ grows exponentially at the rate 2γ.

4. By a change of variables, convert Eq. (1) into a partial differential equation with constant coefficients. Assuming that at $t = 0$, $F(B, 0) = \delta(B - B_0)$, show that the solution of the equation is
$$F(B, t) = \frac{e^{-(9/20)\gamma t}}{B_0 \sqrt{(4/5)\pi \gamma t}} (\frac{B}{B_0})^{1/2} \exp\left\{ -\frac{[\ln(B/B_0)]^2}{(4/5)\gamma t} \right\},$$
so the PDF is a lognormal spreading with time. In your solution, you can use the fact that the diffusion equation
$$\frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2}$$
has the following Green's function solution
$$f(t, z) = \int dz' \frac{f(t', z')}{\sqrt{4\pi D(t-t')}} \exp\left[-\frac{(z-z')^2}{4D(t-t')} \right], \quad t' < t.$$

4 Anisotropic MHD Turbulence

Describe the line of reasoning that leads to the Iroshnikov-Kraichnan $k^{-3/2}$, weak turbulence $k_\perp^{-2}$, and Goldreich-Sridhar $k_\perp^{-5/3}$ spectra of MHD turbulence in the presence of a strong mean field. Be sure to state all assumptions, including those regarding the presence or absence of isotropy and the relative magnitude of the Alfvén and eddy times. Why is the spectrum of MHD turbulence not uniquely determined by purely dimensional considerations — what is the difference with the hydrodynamic case?

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