UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 8 June, 2005 1.30 to 3.30

PAPER 73

ACCRETION DISCS

*Attempt **TWO** questions.*

*There are **THREE** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 The vertical structure of a thin, Keplerian accretion disc with constant (Thomson) opacity and negligible radiation pressure is governed by the equations

$$\frac{\partial p}{\partial z} = -\rho\Omega^2 z, \tag{1}$$

$$\frac{\partial F}{\partial z} = \frac{9}{4}\alpha\Omega p, \tag{2}$$

$$\frac{\partial T}{\partial z} = -\frac{3\kappa\rho F}{16\sigma T^3},$$

$$p = \frac{k\rho T}{\mu_{\mathrm{m}} m_{\mathrm{p}}}.$$

(i) Explain briefly the physical meaning of each equation.

(ii) Show that the problem of the vertical structure of such a disc can be reduced to a universal system of dimensionless ordinary differential equations and boundary conditions by a suitable rescaling of the variables. You may assume that the viscosity parameter α and the mean molecular weight are independent of z and that the 'zero boundary conditions' apply at the surfaces of the disc.

(iii) Assuming that the dimensionless system has a unique solution, deduce that the density-weighted mean kinematic viscosity of the disc is of the form

$$\bar{\nu} = C\alpha^{4/3}(GM)^{-1/3}\left(\frac{\kappa}{\sigma}\right)^{1/3}\left(\frac{\mu_{\mathrm{m}} m_{\mathrm{p}}}{k}\right)^{-4/3} r\Sigma^{2/3},$$

where Σ is the surface density of the disc at a distance r from the central mass M, and C is a dimensionless constant, which need not be determined.

(iv) Equations (1) and (2) involve several approximations that are valid only in the limit of a thin disc. Estimate the order of magnitude of the terms neglected in these equations, and show that the fractional error in each equation is of the order of $(H/r)^2$, where H is the semithickness of the disc.

2 In a frame of reference rotating with uniform angular velocity $\mathbf{\Omega}$, an incompressible fluid of uniform density ρ and kinematic viscosity ν satisfies the equation of motion

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} = -\nabla \psi + \nu \nabla^2 \mathbf{u}.$$

(i) Explain the meaning of the variable ψ, and state a further condition satisfied by the velocity $\mathbf{u}$.

(ii) Let (x, y, z) be Cartesian coordinates in the rotating frame, such that $\mathbf{\Omega} = \Omega\, \mathbf{e}_z$, and let the basic state of the fluid consist of a uniform shear flow $\mathbf{u}_0 = -2Ax\, \mathbf{e}_y$, where A is a constant. Show that exact solutions exist for perturbations of this basic flow, in the form of sheared plane waves such that

$$\mathbf{u} = \mathbf{u}_0 + \mathrm{Re}\left\{\mathbf{v}(t) \exp\left[\mathrm{i}\mathbf{k}(t) \cdot \mathbf{r}\right]\right\}.$$

Obtain the evolutionary equations for $\mathbf{v}(t)$ and $\mathbf{k}(t)$.

(iii) Find the general solution of the evolutionary equations in the case of *two-dimensional* sheared disturbances ($k_z = 0$, $v_z = 0$, but $k_y \neq 0$). Show that the energy of the perturbation is proportional to

$$(1 + T^2)^{-1} \exp\left[-\frac{1}{Re}(T + \tfrac{1}{3}T^3)\right],$$

where $T = 2At$ is a dimensionless time variable with respect to a conveniently chosen origin and $Re = A/(\nu k_y^2)$ is the Reynolds number of the disturbance.

(iv) Determine an approximate expression, valid for $Re \gg 1$, for the maximum factor by which the energy of such a disturbance can be transiently amplified.

(v) Describe in qualitative terms how these results compare with the behaviour of three-dimensional axisymmetric disturbances ($k_y = 0$) in both Keplerian and non-rotating shear flows.

3 Give a summary account of the magnetorotational instability in accretion discs. Your description should incorporate a derivation and analysis of an appropriate dispersion relation such as

$$(\omega^2 - \omega_{\mathrm{A}}^2)^2 - 4\Omega(\Omega - A)\omega^2 - 4\Omega A \omega_{\mathrm{A}}^2 = 0,$$

based on any reasonable simplifying assumptions. Also include a physical interpretation of the mechanism of instability and a brief discussion of its significance for accretion discs.

END OF PAPER